FINITE-DIMENSIONAL DESCRIPTION OF THE LONG-TERM DYNAMICS FOR THE 2D DOI-HESS MODEL FOR LIQUID CRYSTALLINE POLYMERS IN A SHEAR FLOW *

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Abstract. The existence of inertial manifolds for a Smoluchowski equation arising in the 2D Doi-Hess model for liquid crystalline polymers subjected to a shear flow is investigated. The presence of a non-variational drift term complicates the dynamics dramatically from the gradient case in which the it is characterized solely by the steady states. Several transformations are used in order to bring the equation to a form in which the standard theory of inertial manifolds applies. A nonlinear and nonlocal transformation developed in [34] and [35] is used to eliminate the first-order derivative from the micro-micro interaction term. A traveling wave transformation eliminates the first-order derivative from the non-variational term transforming the equation into a nonautonomous one for which the theory of nonautonomous inertial manifolds applies.

Key words. Doi-Hess model; Smoluchowski equation; shear flow; nonautonomous inertial manifolds; Schrödinger-like equation; **subject classifications.** 35Kxx, 70Kxx

1. Introduction

One of the most prominent models for non-Newtonian fluids is the Doi-Hess molecular kinetic theory for nematic liquid crystalline polymers. Polymer molecules are viewed as equal rigid rods (cylinders) of length L and diameter b, $b \ll L$. Their population is then described by a probability distribution function, $f(t,x,\mathbf{m})$, for the axis of symmetry $\mathbf{m} \in S^2$ of the molecule with the center of mass at x at time t. Accounting for Brownian effects leading to rotational and translational diffusion, effects of the flow, and intermolecular interaction, the evolution of f is governed by the so-called Smoluchowski equation

$$\partial_t f + u \cdot \nabla_x f + \nabla_{\mathbf{m}} \cdot (Wf) = D\Delta_x f + D_r \Delta_{\mathbf{m}} f + (D_r/k_B T)\nabla_{\mathbf{m}} \cdot (\nabla_{\mathbf{m}} V f),$$

where $\nabla_{\mathbf{m}} = \mathbf{m} \times \partial_{\mathbf{m}}$ stands for the gradient operator on the unit sphere, and $\Delta_{\mathbf{m}} = \nabla^2_{\mathbf{m}}$ stands for the Laplace-Beltrami operator. The constants c, D_r, D, T and k_B represent the concentration, (pre-averaged) rotational diffusivity, translational diffusivity, absolute temperature T, and the Boltzmann constant k_B, respectively. The equation is also often studied in two dimensions, in which case the orientations $\mathbf{m} \in S^1$, and the above differential operators are modified correspondingly. In the context of nematic polymers, the equation was first proposed in the works of Doi [10] and Hess [20]. It accounts for both the micro-micro interaction between the rods and the macro-micro interaction when the equation coupled to macroscopic fluid equations. If, however, the interaction with the ambient flow is neglected, the equation is a nonlinear Fokker-Planck equation – a gradient system with the free energy as the Lyapunov functional. Historically, the Smoluchowski equation was preceded by a variational model proposed by Onsager in his seminal work [31]. Onsager calculated the free energy functional and derived the Euler-Lagrange equation for the steady-states. He proposed that the interparticle (micro-micro) interaction, due to the excluded volume effects, be modelled using the mean-field ansatz

$$V(t,x,\mathbf{m}) = k_B T \int_0^{2\pi} \beta(\mathbf{m},\mathbf{m}') f(x,\mathbf{m}',t)\; d\mathbf{m}' = \langle \beta(\mathbf{m},\cdot) \rangle_{f(t)}.$$

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The function $\beta(\mathbf{m},\mathbf{m}')$ represents a volume surrounding a molecule with orientation $\mathbf{m}$ within which the center of mass of a molecule with orientation $\mathbf{m}'$ is not allowed. It depends on the shape of the particles, and in the case of cylindric rods it is given by the formula $\beta(\mathbf{m},\mathbf{m}')=2cdL^2|\mathbf{m}\times\mathbf{m}'|$. Expanding β in terms of products of irreducible tensors and retaining only the second term in this expansion, one arrives at the formula $\beta(\mathbf{m},\mathbf{m}')=-3/2U((\mathbf{m}\cdot\mathbf{m}')^2-1/3)$, where $U\propto cL^2b$ denotes the intensity of the potential. Employing this β instead the one proposed by Onsager yields the so-called Maier-Saupe potential (see [29]). This significantly simplifies the mathematical treatment of the model; nevertheless, it is widely accepted that it affords sufficient degrees of freedom to capture the dynamics of the micro-micro interaction. In a recent development, the bifurcation diagram for the Onsager equation (and therefore also Smoluchowski equation in the absence of the flow) with the Maier-Saupe potential was confirmed rigorously (see [6], [7], [9], [13], [25], [26]). The equation undergoes two bifurcations. At a lower potential intensity, the equation undergoes a saddle-node bifurcation, in which a prolate nematic branch of steady-states (probability distribution concentrates to one direction) and an oblate nematic branch of steady-states (probability distribution concentrates uniformly to the equator) emerge. At a higher potential intensity, the equation undergoes a transcritical bifurcation: the oblate branch intersects with the isotropic state, and there is a transfer of stability. Since the Smoluchowski equation in this case is a gradient system, its global attractor is fairly simple: it merely consists of the steady states and their unstable manifolds.

The microscopic Smoluchowski equation is coupled to macroscopic fluid equations (e.g. Navier-Stokes equations) via the drift term

$$W(x,\mathbf{m},t)=(I-\mathbf{mm})(\mathbf{m}\cdot\nabla_x)u$$

and via the viscoelastic stresses that the mesogenic insertions induce in the fluid. There are many challenges to the analysis of the full model, and one resorts to simplifications. Even the simplest situation of spatially homogeneous suspensions in a shear flow leads to complicated and peculiar dynamical behavior; the presence of a non-variational symmetry breaking drift term in the Smoluchowski equation dramatically complicates the dynamics. The equation ceases to be a gradient system, and the attractor becomes a very complicated set, exhibiting not only steady-states, but also various time-periodic solutions (see [12, 17, 18, 24, 27, 30], and even chaos (see [1, 19]). Despite many existing numerical simulations, a rigorous bifurcation analysis presents a great challenge.

Although intrinsically infinite-dimensional, many dissipative parabolic systems exhibit long-term dynamics with properties typical of finite-dimensional dynamical systems. The global attractor, often considered the central object in the study of long-term behavior of dynamical systems, appears to be inadequate in capturing this finite-dimensionality, even when its Hausdorff dimension is finite. This is mainly due to two facts. Firstly, the global attractor can be a very complicated set, not necessarily a manifold; the question whether the dynamics on it can be described by a system of ODEs is yet to be resolved in the literature. Secondly, although all solutions approach this set, they do so at arbitrary rates, algebraic or exponential, and, consequently, the dynamics outside the attractor is not tracked very well on the attractor itself. When they exist, inertial manifolds emerge as most adequate objects to capture the finite-dimensionality of a dissipative parabolic PDE. Introduced by Foias at al. in [15], they are defined to remedy the shortcomings of the global attractor just described: they should be finite dimensional positive-invariant Lipschitz manifolds which attract all solutions exponentially, and on which the solutions of the underlying PDE are recoverable from solutions of a system of ODEs, termed 'inertial form'.

The challenge to proving the existence of inertial manifolds for the Smoluchowski equation lies in the presence of the gradient in the nonlinear Fokker-Planck and the drift terms; in its original form the equation does not satisfy the spectral-gap condition required in all existing theorems for proving the existence of inertial manifolds. The problem was open for a while, and the author recently succeeded in proving their existence for the gradient case by employing a nonlinear nonlocal transformation which eliminates the first-order derivatives from the equation, transforming it into a Schrödinger-like equation. The question remained open for the case in which the Smoluchowski equation is not a gradient system, and for which the question of existence of inertial manifolds becomes even more important due to the complex structure of the global attractor. In this paper, we consider the two-dimensional shear flow case. We employ a similar nonlinear nonlocal transform as the ones in papers [34, 35] to eliminate the gradient from the nonlinear Fokker-Planck term. However, we are still left with gradients in the drift term. The variational portion is then eliminated by another transform of the same kind, while the non-variational portion is eliminated using a traveling wave transformation which transforms the equation to a nonautonomous one. Finally, the spectral gap condition is satisfied, and we apply the theory of nonautonomous inertial manifolds (see [21, 22, 23]), which are now obtained as time-dependent periodic sets. Since the global attractor is completely embedded in the inertial manifold, the dynamics on it is governed by the inertial form. This, at least theoretically at this point, provides an avenue to a rigorous bifurcation analysis of the equation, as well as significant improvement of the existing numerical studies. Some directions are employing the nonlinear Flocket theory, or studying the equation as a nonautonomous perturbation of a gradient system.

Let us remark here that the described method still does not work for the three-dimensional case in the presence of a flow. In the paper [35], the three-dimensional flow in the absence of the flow is treated. However, finding a transformation which eliminates the first-order derivatives from the drift term still presents a challenge.

2. Preliminaries We study a Smoluchowski equation for a spatially homogeneous suspensions of rodlike polymers

$$\partial_t f + \nabla_{\mathbf{m}} \cdot (Wf) = D_r \Delta_{\mathbf{m}} f + (D_r/k_B T)\nabla_{\mathbf{m}} \cdot (\nabla_{\mathbf{m}} V f).$$

The interparticle (micro-micro) interaction term, due to the excluded volume effects, is given by the Maier-Saupe potential

$$V(x,\mathbf{m},t) = k_B T \int_0^{2\pi} \beta(\mathbf{m},\mathbf{m}') f(x,\mathbf{m}',t)\ d\mathbf{m}' = \langle \beta(\mathbf{m},\cdot)\rangle_{f(t)},$$

where $\beta(\mathbf{m},\mathbf{m}') = -3/2U((\mathbf{m}\cdot\mathbf{m}')^2 - 1/3)$ (the 3D case) or $\beta(\mathbf{m},\mathbf{m}') = -2U((\mathbf{m}\cdot\mathbf{m}')^2 - 1/2)$ (2D case). The macro-micro interaction term

$$W(x,\mathbf{m},t) = (I - \mathbf{m}\mathbf{m})(\mathbf{m}\cdot\nabla_x)u$$

is due to the rotation of the axes by the velocity gradients $\nabla_x u$. In this paper, we shall consider spatially homogeneous suspensions in a plane ($f(x,\mathbf{m},t) = f(\mathbf{m},t)$, $x \in \mathbb{R}^2$, $\mathbf{m} \in S^1$) under imposed shear flow $u(x_1,x_2) = (Gx_2, 0)$, where G is the shear rate. We express the particle orientations using a local variable φ, i.e. $\mathbf{m}(\varphi) = (\cos\varphi, \sin\varphi)$, and write $f(\varphi)$ instead of $f(\mathbf{m})$. We will also use the notation $\mathbf{w}(\varphi) = (\cos 2\varphi, \sin 2\varphi)$. The simplest quantity representing its anisotropy of a probability distribution f is the orientational order-parameter tensor which is calculated as the traceless equivalent of the second moment tensor:

$$\mathcal{S}[f] = \langle \mathbf{mm} - \mathrm{I}/2 \rangle_f = \int_0^{2\pi} [\mathbf{m}(\varphi)\mathbf{m}(\varphi) - \mathrm{I}/2] f(\varphi)\ d\varphi.$$

The scalar order parameter

$$S[f] = (2\mathcal{S}[f] : \mathcal{S}[f])^{\frac{1}{2}} = (\langle \mathbf{w} \rangle_f \cdot \langle \mathbf{w} \rangle_f)^{\frac{1}{2}} \in [0,1]$$

represents the degree of molecular alignment. For the isotropic phase $\bar{f} = 1/2\pi$, $S[\bar{f}] = 0$, and for the perfect alignment $S[f] = 1$.

After rescaling, the Smoluchowski equation becomes

$$f_t = f_{\varphi\varphi} + (Wf)_\varphi + (V_\varphi f)_\varphi, \tag{2.1}$$

where the Maier-Saupe potential is given by

$$V[f] = -\frac{U}{2}(\mathbf{mm} - \mathrm{I}/2) : \mathcal{S}[f] = -U \langle \mathbf{w} \rangle_f \cdot \mathbf{w}. \tag{2.2}$$

Here we use the notation $\langle g \rangle_f = \int_0^{2\pi} f(\varphi) g(\varphi)\ d\varphi$. Observe that $|V[f]| \leq U$. Also,

$$W(\varphi) = G \sin^2 \varphi = \frac{G}{2}(1 - \cos 2\varphi).$$

Denoting

$$\tilde{V}(\varphi) = -\frac{G}{4} \sin 2\varphi,$$

the equation (2.1) can be written as

$$f_t - \frac{G}{2} f_\varphi = f_{\varphi\varphi} + ((V_\varphi + \tilde{V}_\varphi) f)_\varphi.$$

Regarding the existence, uniqueness and regularity of solutions of (2.1), it is easy to prove the following theorem (see [6], [7]).

THEOREM 2.1. *Let $f_0 > 0$ be a continuous function on S^1 such that $\int_0^{2\pi} f_0 = 1$. A unique smooth solution $f(t) = S(t) f_0$ of (2.1) for an initial datum $f(0) = f_0$ exists for all nonnegative times, and remains positive and normalized*

$$\int_0^{2\pi} f(\varphi, t)\ d\varphi = 1.$$

Symmetry with respect to the origin – reflecting the fact that that we do not distinguish between orientations $\mathbf{w}$ and $-\mathbf{w}$ – is preserved. Therefore, we can expend the solutions in Fourier series as

$$f(\varphi, t) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} [a_k(t) \cos(2k\varphi) + b_k(t) \sin(2k\varphi)],$$

where

$$a_k(t) = \langle \cos(2k\varphi) \rangle_{f(t)} = \int_0^{2\pi} \cos(2k\varphi) f(\varphi,t)\ d\varphi,$$
$$b_k(t) = \langle \sin(2k\varphi) \rangle_{f(t)} = \int_0^{2\pi} \sin(2k\varphi) f(\varphi,t)\ d\varphi.$$

In this setting, the 2D Smoluchowski equation can also be written as an infinite system of ODEs

$$\begin{aligned} a_0 &= 1 \\ b_0 &= 0 \\ a_k' &= -4k^2 a_k + 2Uk[a_1(a_{k-1} - a_{k+1}) - b_1(b_{k-1} + b_{k+1})] \\ &\quad + \frac{k}{2} G(-b_{k-1} + 2b_k - b_{k+1}) \\ b_k' &= -4k^2 b_k + 2Uk[b_1(a_{k-1} + a_{k+1}) + a_1(b_{k-1} - b_{k+1})] \\ &\quad - \frac{k}{2} G(-a_{k-1} + 2a_k - a_{k+1}) \end{aligned}$$

Multiplying the equation for a_k' by $a_k/2$ and the equation for b_k' by $b_k/2$, and adding the two over $k = 1,2,3,\ldots$ implies the following equation

$$\frac{1}{2}\frac{d}{dt}\sum_{k=1}^{\infty}\frac{a_k^2 + b_k^2}{k} + 4\sum_{k=1}^{\infty} k(a_k^2 + b_k^2) = 2U(a_1^2 + b_1^2) + \frac{G}{2} b_1. \tag{2.3}$$

This implies the dissipativity in the space $H^{-1/2}(S^1)$, and also the fact that if $S[f(t)] \to 0$ as $t \to \infty$ then $f(t) \to \frac{1}{2\pi}$ as $t \to \infty$. Similarly as in [8], one can prove the dissipativity in Gevrey classes of functions, which, in turn, imply the dissipativity in any Sobolev space $H^k(S^1)$. Also, the fact that a_1 and b_1 are determining modes is proven in a similar fashion. Let us state the following

THEOREM 2.2. *The equation (2.1) is dissipative in the Sobolev space $H^k(S^1)$, for any $k \in \mathbb{N}_0$, in the following sense: there exists $\rho_k = \rho_k(U) > 0$ such that for any bounded set $\mathcal{U} \subset L^2(S^1)$ there exist $T_{\mathcal{U}} > 0$ so that for positive $f_0 \in U$ and $t \geq T_{\mathcal{U}}$ the solution $f(t) = S(t) f_0$ satisfies $\|\partial_\varphi^k f(t)\|_{L^2} \leq \rho_k$. In other words, the ball $B^k_{\rho_k} = \{f \in H^k(S^1) : \|f\|_{H^k} \leq \rho_k\}$ is an absorbing set: all solutions of (2.1) enter this set to remain there, eventually. In particular, the solution operator $S(t)$ is compact, and the equation has a finite-dimensional global attractor $\mathcal{A}$. This is the maximal compact set which is invariant: $S(t)\mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{R}$, and attracts all solutions: dist$(S(t)f_0, \mathcal{A}) \to 0$ as $t \to \infty$ for any $f_0 \in L^2(S^1)$.* Let us also remark that the scalar order parameter tensor evolves according to the equation

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt} S[f(t)]^2 &= (2U - 4) S[f(t)]^2 \\ &\quad + 2U(-a_1^2 a_2 - 2a_1 b_1 b_2 + b_1^2 a_2) + \tfrac{G}{2}(b_1 - a_1 b_2 + a_2 b_1) \end{aligned} \tag{2.4}$$

We shall also use the fact that, when $U > 2$, there exists $s_U > 0$ so that for any solution $f(t)$ of the equation (2.1) there exists a time $T_f > 0$ so that $S[f(t)] > s_U$ for all $t \geq T_f$.

3. Inertial Manifolds for Nonautonomous Evolution Equations In this section, we will define inertial manifolds and state a theorem on their existence. As already indicated in the introduction, we are unable to apply the theory of existence of inertial manifolds directly to the Smoluchowski equation – it possesses first-order derivatives in the nonlinear Fokker-Planck and the drift terms, and the equation fails to satisfy the spectral gap condition crucial to the existence of inertial manifolds. For the Smoluchowski equation in the absence of the ambient flow, this difficulty was circumvented by transforming the equation into a form in which it satisfies the spectral gap condition (see [34, 35]). However, in the presence of the flow, in addition to the nonlinear Fokker-Planck term, there is an additional non-variational advection term. On a circle, the equation can again be transformed into a form conducive to the theory; however, the transformed equation turns out to be nonautonomous. Consequently, we study the equation in the context of time-dependent inertial manifolds for nonautonomous evolution equations. As for the autonomous evolution equations, for which the theory originated, there now exists a well developed theory for the nonautonomous evolution equations. Also, there are several approaches to proving their existence. In the papers [21, 22, 23], the authors prove the existence using the cone invariance and the strong squeezing properties, and the theorem applies to our situation.

We consider an evolution equation on a Hilbert space H endowed with the inner product $(\cdot,\cdot)$, and the norm $|\cdot|$ of the form

$$\frac{du}{dt}+Au=N(t,u), \tag{3.1}$$

where A is a positive self-adjoint linear operator with compact inverse, and $N:\mathbb{R}\times H\to H$ is a locally Lipschitz function in u and T-periodic and continuous in t. Recall that, since A^{-1} is compact, there exists a complete set of eigenfunctions w_k for A

$$Aw_k=\lambda_k w_k.$$

We arrange the eigenvalues in a nondecreasing sequence $\lambda_k\le\lambda_{k+1}$, $k=1,2,\dots$ It is a well-known fact that $\lambda_k\to\infty$ as $k\to\infty$. We also define the projection operators

$$P_n u=\sum_{k=1}^{n}(u,w_k)w_k$$

and $Q_n=I-P_n$.

For the autonomous case, i.e., when $N(t,u)=N(u)$, we define inertial manifolds in the following way

Definition 3.1. *An inertial manifold $\mathcal{M}$ is a finite-dimensional Lipschitz manifold which is positively invariant, i.e.*

$$S(t)\mathcal{M}\subset\mathcal{M},\ t\ge 0,$$

and has the exponential tracking property, i.e. there exist $\mu>0$ so that for every $u_0\in H$ there exists $v_0\in\mathcal{M}$

$$|S(t)u_0-S(t)v_0|\le Ke^{-\mu t},\quad t\ge 0$$

where $K=K(u_0,v_0)>0$.

The dynamics of a a nonautonomous system is no longer described by a semigroup, and thus the definitions for invariant sets, global attractor and inertial manifolds have to be modified. Rather than by a semigroup, the dynamics is described by a two parameter family $\{S(t,\theta):t\ge\theta\in\mathbb{R}\}$ (or $S(\cdot,\cdot)$ for short) of continuous operators from H into itself such that

- $S(\theta,\theta)=I$
- $S(t,\sigma)S(\sigma,\theta)=S(t,\theta)$
- $(t,\theta)\mapsto S(t,\theta)u_0$ is continuous for $t\geq\theta$ and $u_0\in H$.

The natural way to define invariance in this context is by the following

Definition 3.2. *A family $A=\{A(t)\}_{t\in\mathbb{R}}$ of nonempty sets $A(t)\subset H$ is called a nonautonomous set. It will be said to be forward invariant if*

$$S(t,\theta)A(\theta)\subset A(t),\ t\geq\theta$$

and invariant if

$$S(t,\theta)A(\theta)=A(t),\ t\geq\theta.$$

In this context, one also talks about two different kinds of attraction (dynamics), the forward ($t\to\infty$), and the pullback ($\theta\to-\infty$) attraction (dynamics). In general, these two dynamics are different, and they will coincide only in some specific situations, e.g. when $S(\cdot,\cdot)$ is a semigroup in disguise. Let us recall here that for autonomous systems, the global attractor can be characterized as the union of all globally in time defined trajectories which are also bounded. In order to achieve the same classification for nonautonomous systems, it turns out that in the definition of the global attractor we need to require the pullback attraction. We arrive at the following

Definition 3.3. *A pullback attractor is defined as an invariant family $A=\{A(t)\}_{t\in\mathbb{R}}$ of compact sets $A(t)\subset H$ such that $\overline{\cup_{t\in\mathbb{R}}A(t)}$ is compact, and it attracts all bounded $B\subset H$ in the pullback sense*

$$\lim_{\theta\to-\infty}\mathrm{dist(S(t,\theta)B,A(t))=0,\ t\in\mathbb{R}}$$

As for inertial manifolds, the definition has to be modified in the following way:

Definition 3.4. *A nonautonomous inertial manifold is a nonautonomous set $\mathcal{M}$ with the properties that*

- *for each $t\in\mathbb{R}$, $\mathcal{M}(t)$is a finite-dimensional Lipschitz manifold;*
- *it is invariant;*
- *it has the exponential tracking property, i.e. there exist $\mu>0$ so that for every $\theta\in\mathbb{R}$ and $u_0\in H$ there exists $v_0\in\mathcal{M}(\theta)$*

$$|S(t,\theta)u_0-S(t,\theta)v_0|\leq Ke^{-\mu t},\quad t\geq 0$$

  where $K=K(\theta,u_0,v_0)>0$.

We shall need the following version of the existence theorem

Theorem 3.1. *Suppose that the nonlinearity $N(t,u)$ in (3.1) satisfies the following three conditions*

- *It has compact support in H, i.e. $supp(N(t,\cdot))\subset B_\rho=\{u\in H:|u|\leq\rho\}$ for some $\rho>0$.*
- *It is continuous in t and $|N(t,u)|\leq C_0$ for $t\in\mathbb{R}$ and $u\in H$ and for some constant $C_0>0$.*
- *It is globally Lipschitz continuous, i.e. $|N(t_1,u_1)-N(t_2,u_2)|\leq C_1|u_1-u_2|+C_2|t_1-t_2|$ for $t_1,t_2\in\mathbb{R}$ and $u_1,u_2\in H$ and for some constants $C_1,C_2>0$.*

Suppose that the eigenvalues of A satisfy the spectral gap condition

$$\lambda_{n+1}-\lambda_n>4C_1,$$

for some $n \in \mathbb{N}$. Then there exists a T-periodic Lipschitz continuous function $\Phi: \mathbb{R} \times P_nH \to Q_nH$ so that the nonautonomous set given as a graph of Φ

$$\mathcal{M} = \mathcal{G}[\Phi] = \{(t, p + \Phi(t,p)) : t \in \mathbb{R}, p \in P_nH\}$$

is an inertial manifold. Restricting (3.1) to $\mathcal{M}$ yields the ordinary differential equation for $p = P_n u$

$$\frac{dp}{dt} + Ap = P_n N(t, p + \Phi(t,p)) \tag{3.2}$$

termed the inertial form. The theory of inertial manifolds is very well established, even for the nonautonomous systems, and there are several different methods for proving their existence, all of them yielding the spectral gap condition as one of the (sufficient) conditions for their existence. In [2], the the authors used the Lyapunov-Perron method to prove the existence of inertial manifolds for the case in which N is assumed to be linear in u and periodic in t. Following this paper, without reproducing the entire proof, we shall indicate here how the inertial manifolds are constructed using the Lyapunov-Perron method for the case of nonlinear N and periodic t.

Proof : Let $\theta \in \mathbb{R}$ and consider the equation

$$\frac{du}{dt} + Au = N(\theta + t, u). \tag{3.3}$$

Let $\beta = \lambda_{n+1} - \lambda_n$ and $\eta = \beta/2$. Define the Banach space

$$X^- = \{f : \mathbb{R}^- \to H : f \text{ continuous and } \sup_{t \le 0} |e^{\eta t} f(t)| < \infty\}$$

First of all, observe that if $u = e^{-\lambda_n t} v$, then (3.3) is equivalent to

$$\frac{dv}{dt} = (\lambda_n - A)v + e^{\lambda_n t} N(t + \theta, e^{-\lambda_n t} v).$$

Next, note that a function $v \in X^-$ is a solution if and only if it satisfies the integral equation

$$\begin{aligned} v(t) = e^{(\lambda_n - A)t} p + \int_0^t e^{(\lambda_n - A)(t-s)} P_n e^{\lambda_n s} N(\theta + s, e^{-\lambda_n s} v(s))\, ds \\ + \int_{-\infty}^t e^{(\lambda_n - A)(t-s)} Q_n e^{\lambda_n s} N(\theta + s, e^{-\lambda_n s} v(s))\, ds, \end{aligned}$$

where $p = P_n v(0)$. To see this, first observe that for a solution $v \in X^-$ and for $\tau \le t \le 0$, we have by variation of constants formula

$$P_n v(t) = e^{(\lambda_n - A)t} p + \int_0^t e^{(\lambda_n - A)(t-s)} P_n e^{\lambda_n s} N(\theta + s, e^{-\lambda_n s} v(s))\, ds$$

and

$$Q_n v(t) = e^{(\lambda_n - A)(t-\tau)} Q_n v(\tau) + \int_\tau^t e^{(\lambda_n - A)(t-s)} Q_n e^{\lambda_n s} N(\theta + s, e^{-\lambda_n s} v(s))\, ds.$$

Then, observe that

$$|e^{(\lambda_n - A)(t-\tau)} Q_n v(\tau)| \le e^{-\beta t + (\beta - \eta)\tau} |v|_{X^-} \to 0$$

as $\tau \to -\infty$. Thus

$$Q_n v(t) = \int_{-\infty}^{t} e^{(\lambda_n - A)(t-s)} Q_n e^{\lambda_n s} N(\theta + s, e^{-\lambda_n s} v(s))\; ds.$$

The converse can also be established using the usual arguments.

Define the function F in the following way:

$$\begin{aligned} F : X^- \times P_n H \times \mathbb{R} &\to X^- \\ (v,p,\theta) &\mapsto e^{(\lambda_n - A)t} p + \int_0^t e^{(\lambda_n - A)(t-s)} P_n e^{\lambda_n s} N(\theta + s, e^{-\lambda_n s} v(s))\; ds \\ &\quad + \int_{-\infty}^{t} e^{(\lambda_n - A)(t-s)} Q_n e^{\lambda_n s} N(\theta + s, e^{-\lambda_n s} v(s))\; ds. \end{aligned}$$

It can be observed easily that F is well defined and that

$$\begin{aligned} |F(v,p,\theta) - F(\bar{v},p,\theta)|_{X^-} &\le C_1 \left(\frac{1}{\eta} + \frac{1}{\beta - \eta} \right) |v - \bar{v}|_{X^-} \\ &\le \frac{4C_1}{\beta} |v - \bar{v}|_{X^-} \end{aligned}$$

The assumption on the spectral gaps insures that F is a contraction with respect to the function v, thus for fixed $p \in P_n H$ and $\theta \in \mathbb{R}$ F possesses a unique fixed point $v(\cdot,p,\theta) \in X^-$. We now define

$$\begin{aligned} \Phi(\theta) : P_n H &\to Q_n H \\ p &\mapsto Q_n v(0,p,\theta) = \int_{-\infty}^{0} e^{sA} Q_n N(\theta + s, e^{-\lambda_n s} v(s))\; ds \end{aligned}$$

The function Φ is T-periodic in θ and Lipschitz continuous in p. To see that, let $v = F(v,p,\theta)$ and $\bar{v} = F(\bar{v},\bar{p},\theta)$ for p and $\bar{p} \in P_n H$. Then

$$\begin{aligned} |v(\cdot,p,\theta) - \bar{v}(\cdot,\bar{p},\theta)|_{X^-} &= |F(v,p,\theta) - F(\bar{v},\bar{p},\theta)|_{X^-} \\ &\le \frac{4C_1}{\beta} |v(\cdot,p,\theta) - \bar{v}(\cdot,\bar{p},\theta)|_{X^-} + |p - \bar{p}| \end{aligned}$$

and so

$$|v(\cdot,p,\theta) - \bar{v}(\cdot,\bar{p},\theta)|_{X^-} \le \frac{\beta}{\beta - 4C_1} |p - \bar{p}|$$

and

$$|\Phi(\theta)(p) - \Phi(\theta)(\bar{p})| \le \frac{4C_1}{\beta - 4C_1} |p - \bar{p}|.$$

□

One of the advantages of having an inertial form for the system is that, at least theoretically, we can preform an asymptotic study of the original system via the Nonlinear Floquet theory. The following theorem can be proven exactly like for its linear counterpart.

Theorem 3.2. *Let $F_{0,t}$ be the the flow map generated by the inertial form (3.2), and let $\Psi = F_{0,T}$ be the corresponding monodromy map. If the monodromy map has* *a logarithm, i.e. if there exists an autonomous vector field Z such that $\Psi=\exp(TZ)$, then there exists T-periodic P so that*

$$F_{0,t}=P(t)\circ\exp(Zt).$$

The mapping P is called the Floquet mapping and the eigenvalues of Z are called the Floquet exponents. Finding the monodromy matrix is a nontrivial task, if at all possible. In the chronological calculus formalism we can write

$$Z(p)=-Ap+\frac{1}{T}\ln\overrightarrow{\exp}\left(\int_0^T P_n N(\tau,p+\Phi(\tau,p)\ d\tau\right).$$

4. The main result

4.1. Schrödinger-like equation In this section we shall transform the Smoluchowski equation in a manner that will eliminate the gradient from the nonlinear term. It can be easily verified that functions f and $V[f]=-U\langle\mathbf{w}\rangle_f\cdot\mathbf{w}$ satisfy (2.1) if and only if

$$u=fe^{V/2}=fe^{-\frac{U}{2}\langle\mathbf{w}\rangle_f\cdot\mathbf{w}} \tag{4.1}$$

and $V[f]$ satisfy the Schrödinger-like equation

$$u_t=u_{\varphi\varphi}+(Wu)_\varphi+\frac{1}{2}\left(V_t+V_{\varphi\varphi}-\frac{1}{2}(V_\varphi)^2-WV_\varphi\right)u.$$

Due to the dependence of V on f, this is not a closed equation in u, and it turns out that the transformation is not invertible. If, however, instead of the minus sign in the exponent in (4.1) we had the plus sign, the transformation would be invertible on an open set of functions. The change of the sign in the exponent of the transformation is easily accomplished if we first preform a transformation which changes the sign of the second modes in the Fourier expansion of f and preserves the positivity of f:

$$g=\Theta(f)=f-2P_2f+d=f+cV+d,$$

where $c=\frac{2}{U\pi}$ and $d=\frac{4}{\pi}$. Notice that $f>0$ implies $g>0$ and $\int_0^{2\pi}f(\varphi)d\varphi=1$ implies $\int_0^{2\pi}g(\varphi)d\varphi=9$. It can be easily seen that f satisfies the Smoluchowski equation if and only if g satisfies

$$g_t=g_{\varphi\varphi}+(Wg)_\varphi+(V_\varphi g)_\varphi+H(g,\varphi), \tag{4.2}$$

where

$$H(g,\varphi)=c[V_t-V_{\varphi\varphi}-(V(W+V_\varphi))_\varphi]-d(W+V_\varphi)_\varphi.$$

As already indicated, the transformation

$$u=g\exp(V/2)=ge^{\frac{U}{2}\langle\mathbf{w}\rangle_g\cdot\mathbf{w}}$$

can be proven to be invertible, and it eliminates the gradient from the nonlinearity. Note that from now on,

$$V[g]=U\langle\mathbf{w}\rangle_g\cdot\mathbf{w}.$$

One can easily verify that the function g satisfies (2.1) if and only if $u=g\exp(V/2)$ satisfies the equation

$$u_t=u_{\varphi\varphi}+(Wu)_\varphi+\frac{1}{2}\left(V_t+V_{\varphi\varphi}-\frac{1}{2}(V_\varphi)^2-WV_\varphi\right)u+H(g)e^{V/2}. \qquad (4.3)$$

In view of the equation for the evolution of $V[g]$ and the fact that $\|V[g]\|_\infty\le U$, we can write the latter equation as

$$u_t=u_{\varphi\varphi}+(Wu)_\varphi+F(g,\varphi), \qquad (4.4)$$

where F depends Lipschitz-continuously on g and φ and is periodic in φ. Our next goal is to express g as a function of u in order to view (4.4) as a closed semilinear parabolic equation in u.

4.2. Transformation inverse The inversion of the transformation requires developing the following framework. For any $u\in L^1(S^1)$, we define the transform $\widehat{u}\in C^\infty(\mathbb{R}^2)$

$$\widehat{u}(\mathbf{x})=\int_0^{2\pi}u(\varphi)e^{-\mathbf{x}\cdot\mathbf{w}(\varphi)}\,d\varphi.$$

Similarly as for the Fourier and the Laplace transforms, for $\mathbf{a}\in\mathbb{R}^2$, we define the operator

$$\mu_{\mathbf{a}}u(\varphi):=u(\varphi)e^{-\mathbf{a}\cdot\mathbf{w}(\varphi)}\in L^1(S^1),$$

and so

$$\widehat{\mu_{\mathbf{a}}u}(\mathbf{x})=\int_0^{2\pi}u(\varphi)e^{-(\mathbf{x}+\mathbf{a})\cdot\mathbf{w}(\varphi)}\,d\varphi=:\tau_{\mathbf{a}}\widehat{u}(\mathbf{x}).$$

We define the function sets $\mathcal{H}=L^2(S^1;\mathbb{R}^+)$, and

$$X=\left\{g\in\mathcal{H}:\int_0^{2\pi}g(\varphi)\,d\varphi=9\right\}.$$

Let also

$$\mathcal{X}=\left\{u\in\mathcal{H}:\int_0^{2\pi}\mu_{\mathbf{a}}u(\varphi)\,d\varphi<9\text{ for some }\mathbf{a}\in\mathbb{R}^2\right\},$$

and $\mathcal{X}_1=\left\{u\in\mathcal{X}:\int_0^{2\pi}u(\varphi)\,d\varphi\ge 9\right\}$, $\mathcal{X}_2=\left\{u\in\mathcal{X}:\int_0^{2\pi}u(\varphi)\,d\varphi<9\right\}$, so that $\mathcal{X}=\mathcal{X}_1\cup\mathcal{X}_2$. For $u\in\mathcal{X}$ we have

$$\nabla\widehat{u}(\mathbf{x})=-\int_0^{2\pi}\mu_{\mathbf{x}}u(\varphi)\mathbf{w}(\varphi)\,d\varphi,$$
$$\nabla\nabla\widehat{u}(\mathbf{x})=\int_0^{2\pi}\mu_{\mathbf{x}}u(\varphi)(\mathbf{w}(\varphi)\mathbf{w}(\varphi))\,d\varphi.$$

$\nabla\nabla\widehat{u}(\mathbf{x})$ is positive definite, since by Cauchy-Schwarz

$$\det(\nabla\nabla\widehat{u})=\langle w_1^2\rangle\langle w_2^2\rangle-\langle w_1w_2\rangle^2>0,$$

where $\langle f\rangle = \int_0^{2\pi} f(\varphi)\mu_x u(\varphi)\ d\varphi$, so $\widehat{u}$ is a concave up.

If $u \in \mathcal{X}_1$, the level set

$$\Gamma(u) = \{\mathbf{x} \in \mathbb{R}^2 | \widehat{u}(\mathbf{x}) \leq 9\}$$

is nonempty convex set, and the there exists a unique point $\mathbf{r} \in \partial\Gamma(u)$ so that $|\mathbf{r}| = \text{dist}(\Gamma(u),\mathbf{o})$, where where $\mathbf{o} = (0,0)$. Note that $\mathbf{r}$ is the unique point on $\partial\Gamma(u)$ for which there exists $U > 0$ such that

$$\nabla\widehat{u}(\mathbf{r}) = -\frac{2}{U}\mathbf{r}.$$

We now define the mappings

$$\begin{aligned}
&R : \mathcal{X}_1 \to \mathbb{R}^2,\\
&\qquad u \mapsto \mathbf{r},\\
&\mathcal{G} : \mathcal{X}_1 \to X,\\
&\qquad u \mapsto g = \mu_{R(u)} u = u e^{-R(u)\cdot\mathbf{w}},\\
&Y : \mathcal{X}_1 \to \mathbb{R}^2,\\
&\qquad u \mapsto -(\nabla\widehat{u})(R(u)) = \int_0^{2\pi} u(\varphi) e^{-R(u)\cdot\mathbf{w}(\varphi)}\mathbf{w}(\varphi)\ d\varphi = <\mathbf{w}>_{\mathcal{G}(u)}\\
&U : \mathcal{X}_1 \to \mathbb{R}^+,\\
&\qquad u \mapsto U = 2|R(u)|/|Y(u)|.
\end{aligned}$$

Note the inequality $R(u) \leq U(u)/2$. We will need the following:

LEMMA 4.1. *R, $\mathcal{G}$, Y, and U are continuous functions on $\mathcal{X}_1$.*

Proof : We prove the continuity of R, and the continuities of $\mathcal{G}$, Y and U follow. To prove the statement by contradiction, we chose a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{X}_1$ and $u \in \mathcal{X}_1$ such that $v_n \to u$ in $L^2(S^2)$. This obviously implies $\widehat{v_n} \to \widehat{u}$ and $\widehat{v_n}' \to \widehat{u}'$ in $L^\infty(S^2)$. Let $\mathbf{r} = R(u)$, $\mathbf{s}_n = R(v_n)$, and suppose $\mathbf{s}_n \not\to \mathbf{r}$ as $n \to \infty$. Let $U_n = 2|R(v_n)|/|Y(v_n)| = 4|\mathbf{s}_n|/|\nabla\widehat{v}_n(\mathbf{s}_n)|$. One can easily observe that the sequence $(\mathbf{s}_n)$ is bounded. Therefore, without loss of generality, we can assume that $\mathbf{s}_n \to \mathbf{s} \neq \mathbf{r}$ as $n \to \infty$. Because of the convergence in the sup norm, $\widehat{v_n}(\mathbf{s}_n) \to \widehat{u}(\mathbf{s})$ and $\nabla\widehat{v_n}(\mathbf{s}_n) \to \nabla\widehat{u}(\mathbf{s})$. Therefore, $\widehat{v_n}(\mathbf{s}_n) = 9$ implies $\widehat{u}(\mathbf{s}) = 9$, and $U_n \to U := 2|\mathbf{s}|/|\nabla\widehat{u}(\mathbf{s})|$, so $\nabla\widehat{u}(\mathbf{s}) = -\frac{2}{U}\mathbf{s}$. This is a contradiction to $\mathbf{s} \neq \mathbf{r}$.

As discussed earlier, $g(t) = \Theta(f(t))$ is a solution of (4.2) for some $U > 0$, if and only if

$$u(t) = g(t)e^{V[g(t)]/2}$$

satisfies

$$u_t = u_{\varphi\varphi} + (Wu)_\varphi + F(\mathcal{G}(u),\varphi). \tag{4.5}$$

As already mentioned, we need Lipschitz continuity of the nonlinear term in order to apply the classical theory of inertial manifolds. In the following lemmas, we shall establish some facts about the Lipschitz continuity of the transformation.

LEMMA 4.2. *Let $U > 0$ be fixed, and let $\mathcal{X}_U = U^{-1}\{U\}$. The functions $R|_{\mathcal{X}_U}$, $\mathcal{G}|_{\mathcal{X}_U}$, and $Y|_{\mathcal{X}_U}$, are Lipschitz continuous. In particular, $\mathcal{G}_U = \mathcal{G}|_{\mathcal{X}_U} : \mathcal{X}_U \to X$ is a Lipschitz homeomorphism. Its inverse is given by*

$$\mathcal{G}_U^{-1}(g) = g e^{(U/2)<\mathbf{w}>_g\cdot\mathbf{w}}.$$

Proof : We prove the Lipschitz continuity of R, and the others follow. Let $u,v \in \mathcal{X}_U$, and let $\mathbf{r} = R(u)$, $\mathbf{s} = R(v)$. The mean-value theorem implies the existence of $\theta_1 \in [0,1]$ and $\theta_2 \in [0,1]$ so that, with the convexity of $\widehat{u}$ and $\widehat{v}$, we have

$$\widehat{u}(\mathbf{s}) - \widehat{u}(\mathbf{r}) = \nabla \widehat{u}(\mathbf{r} + \theta_1(\mathbf{s}-\mathbf{r})) \cdot (\mathbf{s}-\mathbf{r}) \geq \nabla \widehat{u}(\mathbf{r}) \cdot (\mathbf{s}-\mathbf{r}) = -\frac{2}{U}\mathbf{r}(\mathbf{s}-\mathbf{r})$$

and

$$\widehat{v}(\mathbf{r}) - \widehat{v}(\mathbf{s}) = \nabla \widehat{v}(\mathbf{s} + \theta_2(\mathbf{r}-\mathbf{s})) \cdot (\mathbf{r}-\mathbf{s}) \geq \nabla \widehat{v}(\mathbf{s}) \cdot (\mathbf{r}-\mathbf{s}) = -\frac{2}{U}\mathbf{s}(\mathbf{r}-\mathbf{s}).$$

Adding both equations yields

$$\int_0^{2\pi} (u(\varphi) - v(\varphi))(e^{-\mathbf{s}\cdot\mathbf{w}(\varphi)} - e^{-\mathbf{r}\cdot\mathbf{w}(\varphi)})\, d\varphi \geq \frac{2}{U}|\mathbf{s}-\mathbf{r}|^2,$$

and therefore there exists $C_3 = C_3(U) > 0$ so that

$$\frac{2}{U}|\mathbf{s}-\mathbf{r}|^2 \leq C_3 \|u-v\|_{L^2} |\mathbf{r}-\mathbf{s}|,$$

and so $|R(u) - R(v)| \leq \frac{UC_3}{2}\|u-v\|_{L^2}$. □

For $\kappa > 0$ let us define the ball $\mathcal{B}_\kappa = \{u \in \mathcal{H} : \|u\|_{L^2} \leq \kappa\}$. Let us now choose $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$, continuous and increasing, such that the ball $\mathcal{B}_{\kappa(U)}$ satisfies $\mathcal{B}_{\kappa(U)} \supset \mathcal{G}_U^{-1}(\Theta(B_{\rho_0(U)}))$. Observe that the ball $\mathcal{B}_{\kappa(U)}$ is an absorbing ball in $L^2(S^1)$ for the transformed equation (4.5).

We have proved the Lipschitz continuity of the transformation on the set of functions $U^{-1}\{U\}$ on which $U(u)$ is kept fixed at U. Since the potential intensity U is given a-priori, and it is not changed by the original equation, this is also true for the transformed equation. However, the transform changes the geometry of the phase space, and $U^{-1}\{U\}$ is not a Hilbert space. In order to apply the classical theory, we need to imbed this set in a larger Hilbert space, and we accomplish this by allowing U to change; $U(u)$ becomes a quantity that is preserved under the solution operator. In the following lemma we expand the already established Lipschitz continuity to this larger Hilbert space.

Lemma 4.3. *Let $U_1 > 0$ and $K > 0$. Let $\mathcal{U} = \mathcal{B}_{\kappa(U_1)} \cap U^{-1}(0,U_1) \cap \{u \in \mathcal{X}_1 : K < |Y(u)|\}$. Then $R|_\mathcal{U}$, $\mathcal{G}|_\mathcal{U}$, $Y|_\mathcal{U}$, $U|_\mathcal{U}$ are Lipschitz continuous.*

Proof : Let $u,v \in \mathcal{U}$, and let $\mathbf{r} = R(u)$, $\mathbf{s} = R(v)$. As before, we have

$$\widehat{u}(\mathbf{s}) - \widehat{u}(\mathbf{r}) \geq \frac{2}{U(u)}\mathbf{r}(\mathbf{r}-\mathbf{s})$$

and

$$\widehat{v}(\mathbf{r}) - \widehat{v}(\mathbf{s}) \geq \frac{2}{U(v)}\mathbf{s}(\mathbf{s}-\mathbf{r}).$$

Since $\mathbf{r}(\mathbf{r}-\mathbf{s}) + \mathbf{s}(\mathbf{s}-\mathbf{r}) = |\mathbf{r}-\mathbf{s}|^2 \geq 0$, we distinguish the following cases:

Case 1: $\mathbf{r}(\mathbf{r}-\mathbf{s}) \geq 0$ and $\mathbf{s}(\mathbf{s}-\mathbf{r}) \geq 0$.

In this case, similarly as in the previous Lemma, we have

$$\int_0^{2\pi} (u(\varphi) - v(\varphi))(e^{-\mathbf{s}\cdot\mathbf{w}(\varphi)} - e^{-\mathbf{r}\cdot\mathbf{w}(\varphi)})\, d\varphi \geq \frac{2}{U_1}|\mathbf{s}-\mathbf{r}|^2,$$

and so $|R(u)-R(v)| \le \frac{U_1 C_3(U_1)}{2}\|u-v\|_{L^2}$.
Case 2: $\mathbf{r}(\mathbf{r}-\mathbf{s})<0$ and $\mathbf{s}\notin\Gamma(u)$.
In this case,

$$\widehat{u}(\mathbf{s})-\widehat{u}(\mathbf{r})>0>\frac{2}{U(v)}\mathbf{r}(\mathbf{r}-\mathbf{s})$$

and

$$\widehat{v}(\mathbf{r})-\widehat{v}(\mathbf{s})\ge\frac{2}{U(v)}\mathbf{s}(\mathbf{s}-\mathbf{r}),$$

and one arrives at the same conclusion as in the previous case.
Case 3: $\mathbf{r}(\mathbf{r}-\mathbf{s})<0$ and $\mathbf{s}\in\Gamma(u)$.
Since $\mathbf{s}\in\Gamma(u)$, there exists $\mathbf{s}'\in\partial\Gamma(u)\cap[\mathbf{o},\mathbf{s}]$. Let $v'=\mu_{\mathbf{s}-\mathbf{s}'}v$, and so $\widehat{v}'=\tau_{\mathbf{s}-\mathbf{s}'}\widehat{v}$. Thus, $\widehat{v}'(\mathbf{s}')=\widehat{v}(\mathbf{s})=9$, so $R(v')=\mathbf{s}'$ follows. Another easy observation is that $\mathbf{r}(\mathbf{r}-\mathbf{s}')\le 0$, and so

$$\widehat{u}(\mathbf{s}')-\widehat{u}(\mathbf{r})=0\ge\frac{2}{U(v)}\mathbf{r}(\mathbf{r}-\mathbf{s}')$$

and

$$\widehat{v}'(\mathbf{r})-\widehat{v}'(\mathbf{s}')\ge\frac{2}{U(v')}\mathbf{s}'(\mathbf{s}'-\mathbf{r})\ge\frac{2}{U(v)}\mathbf{s}'(\mathbf{s}'-\mathbf{r}),$$

and we conclude again $|\mathbf{r}-\mathbf{s}'|\le\frac{U_1C_3(U_1)}{2}\|u-v'\|_{L^2}$. On the other hand, since $\mathbf{s}$ and $\mathbf{s}'$ are collinear,

$$\widehat{v}(\mathbf{s}')-\widehat{v}(\mathbf{s})\ge\frac{2}{U(v)}\mathbf{s}(\mathbf{s}-\mathbf{s}')=\frac{2}{U(v)}|\mathbf{s}||\mathbf{s}-\mathbf{s}'|\ge K|\mathbf{s}-\mathbf{s}'|.$$

Since $\widehat{v}(\mathbf{s})=\widehat{u}(\mathbf{s}')$, we have $|\mathbf{s}-\mathbf{s}'|\le\frac{U_1C_3(U_1)}{K}\|v-u\|_{L^2}$. The desired follows with the estimate

$$\begin{aligned}\|v-v'\|_{L^2}^2&=\int_0^{2\pi}(v(\varphi)-v'(\varphi))^2d\varphi=\int_0^{2\pi}v(\varphi)^2(1-e^{(\mathbf{s}'-\mathbf{s})\cdot\mathbf{w}(\varphi)})^2\,d\varphi\\&=\int_0^{2\pi}[\mathcal{G}(v)(\varphi)]^2(e^{\mathbf{s}\cdot\mathbf{w}(\varphi)}-e^{\mathbf{s}'\cdot\mathbf{w}(\varphi)})^2\,d\varphi\le C_4^2|\mathbf{s}-\mathbf{s}'|^2\|\mathcal{G}(v)\|_{L^2}^2\le C_5^2\|v-u\|_{L^2}^2.\end{aligned}$$

where the constants C_4 and C_5 depend on K and U_1 only.
Case 4: $\mathbf{s}(\mathbf{s}-\mathbf{r})<0$.
The inequalities for this case follow in an analogue fashion to the previous two cases. □

Lemma 4.4. *Let $0<U<U_1$ and $K=s_U$, and let $\mathcal{U}$ be defined as in the previous Lemma. Let $\mathcal{V}=\mathcal{U}\cup(\mathcal{X}_U\cap\mathcal{B}_{\kappa(U)})$. Then the functions $R|_{\mathcal{V}}$, $\mathcal{G}|_{\mathcal{V}}$, $Y|_{\mathcal{V}}$, and $U|_{\mathcal{V}}$ are Lipschitz continuous.*
Proof : Let us partition $\mathcal{V}$ into three regions: $\mathcal{V}_1=\mathcal{X}_U\cap\mathcal{B}_{\kappa(U)}\cap R^{-1}(0,s_1/2]$, $\mathcal{V}_2=\mathcal{X}_U\cap\mathcal{B}_{\kappa(U)}\cap R^{-1}(s_1/2,s_1)$, and $\mathcal{U}$. By the previous two Lemmas, the functions are Lipschitz continuous on any of these three regions, as well as on sets $\mathcal{V}_1\cup\mathcal{V}_2$ and $\mathcal{V}_2\cup\mathcal{U}$. The Lipschitz continuity on $\mathcal{V}_1\cup\mathcal{U}$ follows since both of these sets are bounded in L^2 and $\operatorname{dist}(R(\mathcal{V}_1),R(\mathcal{U}))>s_1/2$. This implies the Lipschitz continuity on $\mathcal{V}$. □

4.3. Prepared equation Before we can prepare the equation, and apply the theory, we need to preform a couple transformations more in order to have the equation in the form to which the classical theory applies. Firstly, we eliminate the variational portion of the drift $\tilde{V} = -\frac{G}{4}\sin 2\varphi$ term by employing the Lipschitz-homeomorphism $u = \mathcal{K}(v) = ve^{-\tilde{V}/2}$. This leads similarly as before to an equation of the form

$$v_t = v_{\varphi\varphi} + \frac{G}{2} v_\varphi + \tilde{F}(\mathcal{G}(\mathcal{K}(v)), \varphi), \tag{4.6}$$

where $\tilde{F}$ is Lipschitz continuous in both components and is periodic in φ. Denoting

$$N(v,\varphi) = \tilde{F}(\mathcal{G}(\mathcal{K}(v), \varphi)$$

we can write the equation as

$$v_t = v_{\varphi\varphi} + \frac{G}{2} v_\varphi + N(v,\varphi). \tag{4.7}$$

We now follow the usual procedure known as preparing the equation in which we modify the nonlinearity outside the absorbing set $\mathcal{K}^{-1}\mathcal{B}_{2\kappa(U_1)}$. We modify the nonlinear term:

$$N_P(v,\varphi) = \begin{cases} N(v,\varphi), & \text{if } \mathcal{K}v \in \mathcal{V} \\ 0, & \text{if } \mathcal{K}v \in \mathcal{H} \setminus \mathcal{B}_{2\kappa(U_1)} \end{cases}$$

This is clearly a Lipschitz function. Denote by $C > 0$ its Lipschitz constant. Following [36], a Lipschitz-continuous function defined on a subset of a Hilbert space can be extended to a Lipschitz continuous function defined on the entire Hilbert space, even preserving the Lipschitz constant $C > 0$. Without changing the notation, let us by $N_P : \mathcal{H} \times \mathbb{R} \to \mathbb{R}$ denote such an extension. The prepared equation reads now

$$v_t = v_{\varphi\varphi} + \frac{G}{2} v_\varphi + N_P(v,\varphi). \tag{4.8}$$

Finally, the traveling wave transformation $w(t,\varphi) = v(t, \varphi - \frac{G}{2}t)$ leads to the equation

$$w_t = w_{\varphi\varphi} + N_P(w, \varphi - \frac{G}{2}t). \tag{4.9}$$

The term $N_P(w, \varphi - \frac{G}{2}t)$ is now globally Lipschitz, periodic in both t and φ, and we find ourselves in the situation of Theorem 3.1.

4.4. Main theorem We are now able to prove the existence of the inertial manifold of the Smoluchowski equation.

THEOREM 4.5. *Let $U > 2$. The Smoluchowski equation (2.1) possesses in inertial manifold $\mathcal{M}_U$.*

Proof : The positivity of $Aw = -w_{\varphi\varphi}$, the global Lipschitz continuity of N_P and the fact that it vanishes outside of a ball in $\mathcal{H}$ suffice to prove that the prepared equation has a solution for all positive times for any initial datum in $\mathcal{H}$ and it is dissipative. The complete set of eigenfunctions for the linear operator A is given by $w_1^n(\varphi) = \cos n\varphi$, $w_2^n(\varphi) = \sin n\varphi$, $n = 0, 1, \ldots$, with eigenvalues $\lambda_n = n^2$, $n = 0, 1, \ldots$. If C_1 is a Lipschitz constant for N_P, there exists $n \in \mathbb{N}$ such that $\lambda_{n+1} - \lambda_n = 2n + 1 > 4C_1$, and the spectral gap condition is satisfied. The Theorem 3.1 applies, and we infer the existence of nonautonomous time-periodic inertial manifold $\mathcal{M}_P$ for the prepared equation (4.9) given as a graph of a Lipschitz function Φ_P:

$$\mathcal{G}[\Phi_P]=\{(t,p+\Phi_P(t,p)):t\in\mathbb{R},p\in P_n\mathcal{H}\}.$$

Because of the fact that Φ_P is Lipschitz in both components, it can be easily seen that the set

$$\mathcal{M}_P=\bigcup_{t\in\mathbb{R}}(p+\Phi_P(t,p))(\cdot+\frac{G}{2}t)$$

is an inertial manifold for (4.8). We now define $\mathcal{M}_U=B_{\rho_0(U)}\cap\Theta^{-1}\mathcal{G}_U(\mathcal{X}_U\cap\mathcal{K}(\mathcal{M}_P))$. Since $\mathcal{G}_U:\mathcal{X}_U\to X$ is a Lipschitz homeomorphism, it is immediate that $\mathcal{M}_U$ is a finite dimensional Lipschitz manifold. It is positively invariant under, since both $B_{\rho_0(U)}$ and $\Theta^{-1}\mathcal{G}_U(\mathcal{X}_U\cap\mathcal{K}(\mathcal{M}_P))$ are positively invariant. It is also nonempty, since it contains the global attractor of (4.8). It remains to prove that $\mathcal{M}_U$ is exponentially tracking. Let $f_0\in H$ and $f(t)=S(t)f_0$. Let $v(t)=\mathcal{K}^{-1}\mathcal{G}_U^{-1}\Theta(f(t))$, $t\geq 0$. Since $\mathcal{M}_P$ is exponentially tracking, there exists $v_0\in\mathcal{M}_P$ so that for the solution v_P of (4.8) to this initial datum we have $\|v(t)-v_P(t)\|_{L^2}\to 0$, as $t\to\infty$, exponentially. Also, there exists $T>0$ so that $v_P(t)\in\mathcal{K}^{-1}(\mathcal{U})$ for $t\geq T$. However, since $N_P|_{\mathcal{K}^{-1}(\mathcal{U})}=N|_{\mathcal{K}^{-1}(\mathcal{U})}$, $U(\mathcal{K}(v_P(t)))=U$ for $t\geq T$. Therefore, $h(t):=\Theta^{-1}\mathcal{G}\mathcal{K}(v_P(t))\in\Theta^{-1}\mathcal{G}\mathcal{K}(\mathcal{M}_P)$, $t\geq T$ is a solution of (2.1). For some $T'\geq T$ we have $h(t)\in B_{\rho_0(U)}$, $t\geq T'$, and therefore $h(t)\in\mathcal{M}_U$, $t\geq T'$. Finally, since all the transformations are Lipschitz continuous, $\|f(t)-h(t)\|_{L^2}\to 0$, as $t\to\infty$, exponentially. This concludes the proof. □

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