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Inertial manifolds for a Smoluchowski equation on a circle

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Abstract

The existence of inertial manifolds for a Smoluchowski equation—a nonlinear and nonlocal Fokker–Planck equation which arises in the modelling of colloidal suspensions—is investigated. The difficulty due to first-order derivatives in the nonlinearity is circumvented by a transformation.

Mathematics Subject Classification: 35Kxx, 70Kxx

1. Introduction

The Smoluchowski equation describes the temporal evolution of the probability distribution function ψ for directions of rod-like particles in a suspension. The equation has the form of a Fokker–Planck equation

$$\partial_t \psi = \Delta\psi + \nabla \cdot (\psi \nabla V).$$

It is, however, phrased on either the unit sphere or the unit circle, and the gradient, the divergence and the Laplacian are correspondingly modified. The unknown function V stands for a mean-field potential resulting from the excluded volume effects due to steric forces between molecules. Unlike the Fokker–Planck equation, the equation is quadratically nonlinear due to the dependence of V on the probability distribution ψ, and it is nonlocal, since this dependence is nonlocal, as well. In this paper, we shall use a particular type of mean-field potential due to Maier and Saupe [22], which can be thought of as the projection of the probability distribution function on the second eigenspace of the Laplace–Beltrami operator multiplied by a constant.

Historically, the Smoluchowski equation was preceded by a variational model for colloidal suspensions due to Onsager [23]. Onsager calculated the free energy functional, and derived the Euler–Lagrange equation for the steady states. The mean-field potential used in his work was different, and the Maier–Saupe potential is a truncation of this potential. However, it is widely accepted that it affords sufficient degrees of freedom to capture the dynamics of the micro–micro interaction. In a recent development, the bifurcation diagram was confirmed rigorously for both the 2D and the 3D cases (see [5, 6, 8, 13, 18, 19]). In the 2D case, as the potential intensity increases, the equation undergoes a pitchfork bifurcation, in which two equivalent nematic steady states (probability distribution concentrates to one direction) emerge from the isotropic one. In the 3D case, the equation undergoes two bifurcations. At a lower potential intensity, the equation undergoes a saddle-node bifurcation, in which a prolate nematic steady state (probability distribution concentrates to one direction) and an oblate nematic steady state (probability distribution concentrates uniformly to the equator) emerge.

The Smoluchowski equation was first proposed in the works of Doi [10] and Hess [17] as a dynamical model for nematic liquid crystalline polymers. In this paper, we study a version of the equation in which the interaction with the ambient flow is neglected. In this case, the equation becomes a gradient system with free energy as the Lyapunov functional, it is dissipative in a Gevrey class of analytic functions and it possesses a finite-dimensional global attractor consisting of the steady states and their unstable manifolds. In addition, the second eigenmode is a single determining mode for the 2D equation. All these facts point to the finite dimensionality of the dynamics. The best framework for describing the finite dimensionality of a partial differential equation (PDE) is within the context of inertial manifolds, when they exist. First introduced by Foias *et al* in [15], they are defined as finite-dimensional positive-invariant Lipschitz manifolds which attract all solutions exponentially, and on which the solutions of the PDE can be recovered from solutions of a system of ordinary differential equations (ODEs), termed the inertial form. One of the most notable examples of parabolic PDEs which possess inertial manifolds is the Kuramoto–Sivashinsky equation [14]. There is also a large class of dissipative PDEs, most notably the 2D Navier–Stokes equation, which possess a finite-dimensional attractor, for which, however, the existence of inertial manifolds is still open. The main difficulty in proving the existence of inertial manifolds lies in a very restrictive spectral gap condition; this is especially true in the presence of first-order derivatives in the nonlinearity, as is the case for the Smoluchowski equation and the Navier–Stokes equations. This difficulty for the 2D Smoluchowski equation is circumvented in this paper by transforming it to a parabolic equation that does not contain a spatial derivative in the nonlinearity.

The paper is structured as follows. We first review some basic properties of the 2D Smoluchowski equation. The equation is transformed into an infinite system of ODEs, and the existence of an absorbing cone is proven. In the section that follows, we define inertial manifolds and state a theorem for their existence. Different proofs are available in the literature (e.g. [3, 24, 25]). We then transform the Smoluchowski equation in order to remove the first-order derivatives from the nonlinearity. Since Lipschitz continuity of the nonlinearity is required in order to apply the existence theorem, we then modify the equation outside the absorbing cone, and apply the existing theory of inertial manifolds to this prepared equation. This, in turn, yields the existence of inertial manifolds for the Smoluchowski equation.

Let us also remark that the dynamics of the Smoluchowski equation becomes much more complex when we allow for interaction with the ambient flow. Even a passive interaction with a shear flow leads to complicated and peculiar dynamical behaviour. This is due to the fact that the fluid introduces a nonvariational element to the dynamics, even though the nonlinearity remains unchanged. The equation ceases to be a gradient system, and the attractor becomes a much more complicated object: in addition to flow-aligning (steady states), different time-periodic solution regimes and chaos were confirmed numerically. The method developed here can be modified to prove the existence of inertial manifolds in this dynamically more interesting case (see [26]).

2. Preliminary information

2.1. 2D Smoluchowski equation

We study a Smoluchowski equation on a circle $S^1 = [0, 2\pi]$

$$\psi_t = \psi_{\varphi\varphi} + (\psi V_\varphi)_\varphi. \tag{2.1}$$

The unknown ψ in the equation is the probability distribution function for the orientation of rigid rod-like molecules in a suspension, and the unknown V is a mean-field intermolecular interaction potential resulting from the excluded volume effects due to steric forces between molecules. In this paper, we use the Maier–Saupe potential given by

$$V(\varphi, t) = -b(\boldsymbol{m} \otimes \boldsymbol{m} - I/2) : \langle \boldsymbol{m} \otimes \boldsymbol{m} - I/2\rangle_{\psi(t)}, \tag{2.2}$$

where $\boldsymbol{m}(\varphi) = (\cos\varphi, \sin\varphi)$, $\langle f\rangle_\psi = \int_0^{2\pi} f(\varphi)\psi(\varphi)\,\mathrm{d}\varphi$, and the parameter $b > 0$ denotes the potential intensity. Regarding the existence, uniqueness and regularity of solutions of (2.1), it is easy to prove the following theorem (see [5,6]).

Theorem 1. *Let $\psi_0 > 0$ be a continuous function on S^1 such that $\int_0^{2\pi} \psi = 1$. A unique smooth solution $\psi(t) = S(t)\psi_0$ of (2.1) for an initial datum $\psi(0) = \psi_0$ exists for all nonnegative times, and remains positive and normalized*

$$\int_0^{2\pi} \psi(\varphi, t)\,\mathrm{d}\varphi = 1.$$

The Smoluchowski equation preserves certain symmetries. Symmetry with respect to the origin is preserved, reflecting the fact that we do not distinguish between orientations $\boldsymbol{m}$ and $-\boldsymbol{m}$. Also, symmetry with respect to any line passing through the origin is preserved. Therefore, the form of a solution ψ expended in a Fourier series as

$$\psi(\varphi, t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} y_k(t)\cos(2k\varphi),$$

where

$$y_k(t) = \langle \cos 2k\varphi\rangle_\psi = \int_0^{2\pi} \cos(2k\varphi)\psi(\varphi, t)\,\mathrm{d}\varphi$$

is preserved. We will restrict ourselves to the study of solutions with such symmetry. The normalization implies $y_0 = 1$ and $|y_k| \leqslant 1$. In this setting, the 2D Smoluchowski equation can also be written as an infinite system of ODEs

$$\begin{aligned} &y_0 = 1, \\ &y_k' + 4k^2 y_k = bky_1(y_{k-1} - y_{k+1}), \qquad k = 1, 2, \ldots \end{aligned} \tag{2.3}$$

In particular,

$$y_1' = [b(1 - y_2) - 4]y_1, \tag{2.4}$$

so the sign of y_1 does not change in the evolution. It is an easy observation that if

$$\psi(\varphi, t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} y_k(t)\cos(2k\varphi)$$

is a solution, so is

$$\psi(\varphi - \pi/2, t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} (-1)^k y_k(t)\cos(2k\varphi).$$

Therefore, every solution ψ with $y_1(t) > 0$, $t \geqslant 0$ has a corresponding solution $\psi(\cdot - \pi/2)$ with $y_1(t) < 0$, $t \geqslant 0$, and vice versa. Also, $y_1 = 0$ ($V \equiv 0$) is preserved by the flow. The solutions on that subspace satisfy the heat equation $\psi_t = \psi_{\varphi\varphi}$, and, consequently, decay exponentially to the steady state $\bar{\psi} = 1/2\pi$. Therefore, it suffices to study the equation for $y_1 > 0$. Note that the potential can be written as

$$V(\varphi, t) = -\frac{b}{2} y_1(t)\cos(2\varphi),$$

and its evolution follows the equation:

$$V_t = [b(1 - y_2) - 4]V. \tag{2.5}$$

2.2. Dissipativity and the global attractor

Let

$$H = \left\{\psi \in L^2(S^1; \mathbb{R}^+) : \psi(-\varphi) = \psi(\varphi), \psi(\varphi + \pi/2) = \psi(\varphi), \int_0^{2\pi} \psi = 1, \langle\cos 2\varphi\rangle_\psi > 0\right\}.$$

For $f(k) = a^{2k}$, $1 < a^2 < 1 + 1/b$, we define the following Gevrey class of functions:

$$H_f := \left\{\psi(\varphi) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} y_k \cos(2k\varphi) : \sum_{k=1}^{\infty} \frac{f(k)}{k} y_k^2 < \infty, \psi > 0, y_1 > 0\right\} \tag{2.6}$$

endowed with the norm

$$|\psi - \bar{\psi}|_f = \left(\sum_{k=1}^{\infty} \frac{f(k)}{k} y_k^2\right)^{1/2}. \tag{2.7}$$

H_f is a subset of the set of real-analytic functions. Also, for each $n \in \mathbb{N} \cup \{0\}$ there exists a combinatorial constant $M_n > 0$ depending on a, such that

$$\|\partial_\varphi^n(\psi - \bar{\psi})\|_\infty \leqslant M_n |\psi - \bar{\psi}|_f, \qquad \psi \in H_f. \tag{2.8}$$

In paper [7], it was proven that (2.1) is dissipative in the sense that, for initial data of the form

$$\psi_0(\varphi) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} y_k(0)\cos(2k\varphi) \tag{2.9}$$

belonging to a bounded set $U \subset H^{-1/2}(S^1)$, there exist $T_U > 0$ so that for $t \geqslant T_U$ the solution satisfies $|\psi(t) - \bar{\psi}|_f \leqslant \sqrt{b}$, and, consequently, $\|\partial_\varphi^n(\psi - \psi_u)\|_\infty \leqslant M_n\sqrt{b}$. In other words, the ball of radius $\sqrt{b}$ in H_f is an absorbing set: all solutions of (2.1) enter this set to remain there, eventually. The Smoluchowski equation possesses even smaller absorbing sets. First, we observe that the quotient $z_k = y_k/y_1$ satisfies the ODE

$$z_k' + 4(k^2 - 1)z_k = bky_1(z_{k-1} - z_{k+1}) - bz_k(1 - y_2), \qquad k = 2, 3, 4, \ldots,$$

and therefore

$$\frac{\mathrm{d}}{2\,\mathrm{d}t}(z_k^2) + 4(k^2 - 1)z_k^2 = bky_1(z_{k-1}z_k - z_k z_{k+1}) - bz_k^2(1 - y_2), \qquad k = 2, 3, 4, \ldots.$$

Multiplying by $f(k)/k$ and summing yields the inequality

$$\frac{\mathrm{d}}{2\,\mathrm{d}t}\sum_{k=2}^{\infty}\frac{f(k)}{k}z_k^2 + 2\sum_{k=2}^{\infty} kf(k)z_k^2 \leqslant b|y_2|.$$

In particular, $|\psi(t)|_f/y_1$ with $\psi_0 \in U$ is dissipated in time, until eventually

$$|\psi(t)|_f < \sqrt{b}y_1, \qquad t \geqslant T_U. \tag{2.10}$$

In particular, the cone-like set

$$C_b = \left\{\psi \in H_f : |\psi|_f < \sqrt{b}y_1\right\}$$

is absorbing and invariant:

$$S(t)C_b \subset C_b, \qquad t \geqslant 0.$$

The eigenmode y_1 is significant also in the sense that it is a determining mode, i.e. if for two solutions

$$\lim_{t\to\infty} |y_1^{(1)}(t) - y_1^{(2)}(t)| = 0,$$

then

$$\lim_{t\to\infty} |\psi^{(1)}(t) - \psi^{(2)}(t)|_f = 0.$$

The 2D Smoluchowski equation has a global attractor $\mathcal{A}$ defined as the maximal compact set which is invariant: $S(t)\mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{R}$, and attracts all solutions: $\mathrm{dist}(S(t)\psi_0, \mathcal{A}) \to 0$ as $t \to \infty$ for any ψ_0. It is easily proven that $\mathcal{A}$ is finite dimensional. Also, the structure of $\mathcal{A}$ is somewhat simple due to the fact that the equation (2.1) is a gradient system. Denoting $u = \psi \exp(V/2)$, the Lyapunov functional is given by

$$\mathcal{F}(t) = \int_0^{2\pi} \psi \log u \, \mathrm{d}\varphi,$$

which satisfies the equation

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = -\int_0^{2\pi} |(V + \log\psi)_\varphi|^2 \psi \, \mathrm{d}\varphi.$$

For dissipative gradient systems, the dynamical behaviour is characterized by a global attractor which is formed by the steady states and their unstable manifolds. The right-hand side of the energy dissipation equation yields the equation for the steady states

$$V + \log\psi = \text{const.},$$

which after denoting $r = (b/4)\langle\cos 2\varphi\rangle_\psi$ can be written as

$$\psi(\varphi) = \frac{\exp(2r\cos 2\varphi)}{\int_0^{2\pi}\exp(2r\cos 2\varphi)}.$$

It was shown that at $b = 4$ equation (2.1) undergoes a pitchfork bifurcation. For $b < 4$ the isotropic solution $\bar{\psi} = 1/2\pi$ is the unique steady state, and it is asymptotically stable. In this case, $\mathcal{A} = \{\bar{\psi}\}$, and all solutions converge to the isotropic steady state, exponentially. For $b > 4$, two anisotropic steady states $\psi_b^\pm$ of the above form ($r = \pm r_b^*$) emerge from the isotropic steady state. Note that the relationship between the two anisotropic steady states is $\psi_b^-(\varphi) = \psi_b^+(\varphi - \pi/2)$. All solutions of the equation (2.1) converge to one of the three steady states. In view of (2.4), if $\psi(t) \to \bar{\psi}$ with $y_1 \not\equiv 0$, then $y_2(t) \to 0$ implies that $y_1(t) \to \infty$ exponentially as $t \to \infty$, which is a contradiction. Therefore, if $b > 4$, $\bar{\psi}$ is a saddle with the set $\{y_1 = 0\}$ as the basin of attraction. The nematic steady states ψ_b^+ and ψ_b^- are attracting points with the sets $\{y_1 > 0\}$ and $\{y_1 < 0\}$, respectively, as basins of attraction. The global attractor consists of the three steady states and the unstable manifolds of the saddle. Since the dynamics on the set $\{y_1 < 0\}$ merely mirrors the dynamics on the set $\{y_1 > 0\}$, and the dynamics on the set $\{y_1 = 0\}$ is trivial, we restrict our study to the set $\{y_1 > 0\}$. In this case, $\mathcal{A} = \{\psi_b^+\}$. However, since $y_2(t) \to 1 - 4/b$ as $t \to \infty$, in view of (2.4), the rates of convergence to ψ_b^+ are hard to determine.

2.3. Inertial manifolds

The Smoluchowski equation possesses a finite-dimensional global attractor, has a single determining mode and has a simple bifurcation structure of a kind which is typically encountered in systems of ODEs. Therefore, even though as a PDE the Smoluchowski equation is intrinsically infinite dimensional, its dynamics exhibits properties typical of finite-dimensional dynamical systems.

The global attractor is a central object in the study of long-term dynamics. It, however, appears to be inadequate in capturing the finite dimensionality of the dynamics. This is mainly due to two facts. Firstly, it can be a very complicated set which is not a manifold, and the dynamics on it may not be reduced to a finite system of ODEs. Secondly, although all solutions approach this set, they do so at arbitrary rates (algebraic or exponential), so the dynamics outside of the attractor is not best tracked on the attractor itself. When they exist, the so-called inertial manifolds are more suitable to capture the finite dimensionality of a PDE. Introduced by Foias *et al* in [15], it was defined to remedy these shortcomings of the global attractor.

Consider an evolution equation on a Hilbert space H endowed with the inner product $(\cdot,\cdot)$, and the norm $|\cdot|$ of the form

$$\partial_t u + Au = N(u), \tag{2.11}$$

where A is a positive self-adjoint linear operator with compact inverse, and $N : H \to H$ is a locally Lipschitz function. Recall that, since A^{-1} is compact, there exists a complete set of eigenfunctions w_k for A

$$Aw_k = \lambda_k w_k.$$

We arrange the eigenvalues in a nondecreasing sequence $\lambda_k \leqslant \lambda_{k+1}$, $k = 1, 2, \ldots$ It is a well-known fact that $\lambda_k \to \infty$ as $k \to \infty$. We also define the projection operators

$$P_n u = \sum_{k=1}^{n} (u, w_k) w_k$$

and $Q_n = I - P_n$, and the cone-like sets

$$\mathcal{C}_l^n = \{w \in H : |Q_n w| \leqslant l |P_n w|\}.$$

Definition 1. *An inertial manifold $\mathcal{M}$ is a finite-dimensional Lipschitz manifold which is positively invariant*

$$S(t)\mathcal{M} \subset \mathcal{M}, \qquad t \geqslant 0,$$

and exponentially attracts all orbits of the flow uniformly on any bounded set $U \subset H$ of initial data

$$\mathrm{dist}(S(t)u_0, \mathcal{M}) \leqslant C_U \mathrm{e}^{-\mu t}, \qquad u_0 \in U,\ t \geqslant 0.$$

The inertial manifold is said to be asymptotically complete if for any solution $u(t)$, there exists $v_0 \in \mathcal{M}$ such that

$$|u(t) - S(t)v_0| \to 0, \qquad t \to \infty,$$

exponentially.

There are several methods for proving the existence of inertial manifolds. The vast majority of them require some kind of Lipschitz continuity of the nonlinearity N and make use of a very restrictive spectral gap property of the linear operator A which is usually the Laplacian. These two conditions yield the strong squeezing property, which, in turn, yields the existence of an inertial manifold. The inertial manifold is obtained as a graph of a Lipschitz mapping.

Theorem 2. *Suppose that the nonlinearity in (2.11) satisfies the following three conditions:*

- *It has compact support in H, i.e.* $\mathrm{supp}(N) \subset B_\rho = \{u \in H : |u| \leqslant \rho\}$
- *It is bounded, i.e.* $|N(u)| \leqslant C_0$ *for* $u \in H$
- *It is globally Lipschitz continuous, i.e.* $|N(u_1) - N(u_2)| \leqslant C_1|u_1 - u_2|$ *for* $u_1, u_2 \in H$.

Suppose that the eigenvalues of A satisfy the spectral gap condition, i.e.

$$\lambda_{n+1} - \lambda_n > 4C_1,$$

for some $n \in \mathbb{N}$. *Then the strong squeezing property holds, i.e.*

- *If* $u_1(0) - u_2(0) \in \mathcal{C}_l^n$, *then* $u_1(t) - u_2(t) \in \mathcal{C}_l^n$ *for* $t \geqslant 0$.
- *If* $u_1(t_0) - u_2(t_0) \notin \mathcal{C}_l^n$, *for some* $t_0 \geqslant 0$, *then* $|Q_n(u_1(t) - u_2(t))| \leqslant |Q_n(u_1(0) - u_2(0))|\mathrm{e}^{-\mu t}$ *for some* $\mu > 0$ *and for* $0 \leqslant t \leqslant t_0$.

The strong squeezing property implies the existence of an asymptotically complete inertial manifold which is the graph of a Lipschitz function $\Phi : P_n H \to Q_n H$, *i.e.*

$$\mathcal{M} = \mathcal{G}[\Phi] = \{p + \Phi(p) : p \in P_n H\}$$

with

$$|\Phi(p_1) - \Phi(p_2)| \leqslant l|p_1 - p_2|.$$

Restricting (2.11) to $\mathcal{M}$ *yields the ODE for* $p = P_n u$

$$\frac{\mathrm{d}p}{\mathrm{d}t} + Ap = P_n N(p + \Phi(p))$$

termed the inertial form.

Different proofs are available in the literature (e.g. [3, 24, 25]). The above result is not the strongest possible. It is possible to ease the Lipschitz condition to allow for nonlinearities that contain first-order derivatives of u, resulting, however, in an even more restrictive spectral gap condition. If we apply this result to the Smoluchowski equation (2.1), it turns out that the spectral gap condition holds only for the intensities $b < 4$, in which case $\mathcal{M} = \mathcal{A} = \{\bar{\psi}\}$. The main idea of this paper is to eliminate the gradient from the nonlinearity of (2.1) through the transformation $u = \psi \exp(V/2)$, and then to apply theorem 2.

3. The main result

3.1. Transformed equation

It can be easily verified that if a normalized, positive function ψ satisfies (2.1), then $u = \psi \exp(V/2) = \psi \exp(-b\langle\cos 2\varphi\rangle_\psi \cos 2\varphi/4)$ satisfies the equation

$$u_t = u_{\varphi\varphi} + \tfrac{1}{2}\left(V_t + V_{\varphi\varphi} - \tfrac{1}{2}(V_\varphi)^2\right)u. \tag{3.12}$$

Equation (2.5) for the evolution of V and the fact that $V_{\varphi\varphi} = -4V$ allow us to rewrite this equation in the form

$$u_t = u_{\varphi\varphi} + F[\psi]u, \tag{3.13}$$

where

$$F[\psi] = -\frac{b\langle\cos 2\varphi\rangle_\psi \cos 2\varphi}{4}\left[b(1 - \langle\cos 4\varphi\rangle_\psi) - 8\right] - \frac{b^2\langle\cos 2\varphi\rangle_\psi{}^2 \sin^2 2\varphi}{4}.$$

Observe that F depends only on the first two eigenmodes of ψ. Our next goal is to express ψ as a function of u in order to view (3.13) as a closed equation in u. To this end, we develop the following framework. If $u \in L^1(S^1)$, we define the transform

$$\hat{u}(x) = \int_0^{2\pi} u(\varphi)\mathrm{e}^{x\cos 2\varphi}\,\mathrm{d}\varphi.$$

Clearly, $\hat{u} \in C^\infty(\mathbb{R})$. For $a \in \mathbb{R}$ we define

$$m_a u(\varphi) := u(\varphi)\mathrm{e}^{a\cos 2\varphi} \in L^1,$$

and so

$$\widehat{m_a u}(x) = \int_0^{2\pi} u(\varphi)\mathrm{e}^{(x+a)\cos 2\varphi}\,\mathrm{d}\varphi =: \tau_a \hat{u}(x).$$

For $u = 1/2\pi$ let us denote

$$h(x) =: \hat{u}(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{x\cos 2\varphi}\,\mathrm{d}\varphi.$$

Note that since

$$h^{(n)}(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos^n(2\varphi)\mathrm{e}^{x\cos 2\varphi}\,\mathrm{d}\varphi$$

$h^{(n)} > 0$ for n even and since $h^{(n)}(0) = 0$ for n odd, we also have $h^{(n)}(x) > 0$ for $x > 0$ and n odd. Remark also that $h''(0) = 1/2$.

We define the set

$$\mathcal{X} = \cup_{a\in\mathbb{R}} \left\{ m_a v : v \in L^2(S^1; \mathbb{R}^+),\, v \text{ even}, \int_0^{2\pi} v(\varphi)\,\mathrm{d}\varphi < 1 \right\}.$$

Obviously, $\mathcal{X}$ is an open subset of $L^2(S^1)$, and for $u \in \mathcal{X}$ we have

$$\hat{u}'(x) = \int_0^{2\pi} u(\varphi)\cos(2\varphi)\mathrm{e}^{x\cos 2\varphi}\,\mathrm{d}\varphi,$$
$$\hat{u}''(x) = \int_0^{2\pi} u(\varphi)\cos^2(2\varphi)\mathrm{e}^{x\cos 2\varphi}\,\mathrm{d}\varphi > 0.$$

Also, if $u \in \mathcal{X}$, and $u = m_a v$ for some a, then $\hat{u}(-a) < 1$. Since $\hat{u}$ is concave up, and $\hat{u}(x) \to \infty$ as $x \to \infty$, there exists a unique $r \in \mathbb{R}$ such that $\hat{u}(r) = 1$ and $\hat{u}'(r) > 0$. We now define the mappings

$$\begin{aligned}
R &: \mathcal{X} \to \mathbb{R}, \\
&\quad u \mapsto r, \\
\Psi &: \mathcal{X} \to H, \\
&\quad u \mapsto \psi = m_{R(u)}u = u\mathrm{e}^{R(u)\cos 2\varphi}, \\
Y_1 &: \mathcal{X} \to \mathbb{R}^+, \\
&\quad u \mapsto \hat{u}'(R(u)) = \int_0^{2\pi} u(\varphi)\mathrm{e}^{R(u)\cos 2\varphi}\cos 2\varphi\,\mathrm{d}\varphi = \langle \cos 2\varphi\rangle_{\Psi(u)}, \\
Y_2 &: \mathcal{X} \to \mathbb{R}, \\
&\quad u \mapsto \int_0^{2\pi} u(\varphi)\mathrm{e}^{R(u)\cos 2\varphi}\cos 4\varphi\,\mathrm{d}\varphi = \langle \cos 4\varphi\rangle_{\Psi(u)},
\end{aligned}$$

and $B = 4R/Y_1$. We will need the following result:

Lemma 1. *R, Ψ, Y_1, Y_2, and B are continuous functions on $\mathcal{X}$.*

Proof. It is enough to prove the continuity of R. First, let us observe that for $u, v \in \mathcal{X}$ and $r = R(u)$, $s = R(v)$ with $s < r$, since $\hat{v}$ is increasing on $[s, \infty)$, we have $\hat{v}(r) > \hat{v}(s) = 1 = \hat{u}(r)$. Since $\hat{v}$ is concave up, we have

$$0 < \hat{v}'(s) \leqslant \frac{\hat{v}(r) - \hat{u}(r)}{r - s}.$$

Thus,

$$r - s \leqslant \frac{1}{\hat{v}'(s)} \int_0^{2\pi} (v(\varphi) - u(\varphi))\mathrm{e}^{r\cos 2\varphi}\,\mathrm{d}\varphi \leqslant \frac{C(b)}{\hat{v}'(s)} \|u - v\|_2, \tag{3.14}$$

and the continuity at v is obvious as long as $s < r$. To prove the other direction, let $v_n \to u$ in L^2. This obviously implies $\widehat{v_n} \to \widehat{u_n}$ and $\widehat{v_n}' \to \widehat{u_n}'$ in L^∞. Let $r = R(u)$, $s_n = R(v_n)$. Due to the above remark, we assume, without loss of generality, that $s_n < r$ for all $n \in \mathbb{N}$. In order to prove the continuity by contradiction we assume that $s_n \not\to r$. Without loss of generality, we can assume that $s_n \uparrow s_0 < r$. The above inequality implies

$$0 < \widehat{v_n}'(s_n) \leqslant \frac{\widehat{v_n}(r) - \hat{u}(r)}{r - s_n} \to 0,$$

so $\widehat{v_n}'(s_n) \to 0$. Since $\widehat{v_n}'(s_n) \leqslant \widehat{v_n}'(s_0) \to \hat{u}'(s_0)$ we have $\hat{u}'(s_0) \geqslant 0$, and therefore $\hat{u}(s_0) < \hat{u}(r) = 0$. This and the fact that $\widehat{v_n}(s_0) \geqslant \widehat{v_n}(s_n) = 0$ yield a contradiction to the fact that $\widehat{v_n}(s_0) \to \hat{u}(s_0)$. □

Corollary 1. *Let $b > 0$ be fixed, and let $\mathcal{X}_b = B^{-1}\{b\}$. The function*

$$\Psi : \mathcal{X}_b \to H$$

is a homeomorphism, where the inverse is given by

$$\Psi^{-1}(\psi)(\varphi) = \psi(\varphi)\mathrm{e}^{-b\langle\cos 2\varphi\rangle_\psi \cos 2\varphi/4}.$$

As already mentioned, if a positive, normalized function ψ is a solution of (2.1) for some $b > 0$, then

$$u(\varphi) = \psi(\varphi)\mathrm{e}^{V(\varphi)/2} = \psi(\varphi)\mathrm{e}^{-by_1\cos 2\varphi/4}$$

satisfies (3.13). Denoting by $r = b\langle\cos 2\varphi\rangle_\psi/4 > 0$, we have $\psi = u\exp(r\cos 2\varphi)$. Then, $\int_0^{2\pi}\psi\,\mathrm{d}\varphi = 1$ implies $\hat{u}(r) = 1$, and multiplying by $\cos 2\varphi$ and integrating, we obtain $\hat{u}'(r) = 4r/b > 0$. Using the framework developed earlier, $r = R(u)$, $y_1 = Y_1(u)$, $y_2 = Y_2(u)$, $\psi = \Psi(u)$ and $b = B(u)$. Thus,

$$F[\Psi(u)] = -R(u)\,[B(u)(1 - Y_2(u)) - 8]\cos 2\varphi - 4R(u)^2\sin^2 2\varphi.$$

Therefore, u satisfies the closed equation

$$u_t = u_{\varphi\varphi} + F[\Psi(u)]u. \tag{3.15}$$

On the other hand, if $u \in \mathcal{X}$ satisfies (3.15), it is immediate that $\Psi(u)$ satisfies (2.1), where $b = B(u) = B(u_0)$ is a quantity preserved by the flow.

3.2. Prepared equation

In order to apply the classical theory, we need the nonlinear term in (3.15) to satisfy a Lipschitz condition. Since this is not necessarily true, we have to modify equation (3.15) in a way that preserves its long-term behaviour, i.e. that does not change the attractor nor the convergence to the attractor. Usually, this is done by modifying the nonlinear term outside of an absorbing set. Let

$$\mathcal{C} = \left\{u \in \Psi^{-1}(H_f) : |\Psi(u)|_f < \sqrt{B(u)}Y_1(u)\right\},$$

and let $\mathcal{C}_b = \mathcal{C} \cap \mathcal{X}_b$, where H_f and $|\cdot|_f$ are defined in (2.6) and (2.7).

Lemma 2. *Let $0 < b \neq 8$. The functions $R|_{\mathcal{C}_b}$, $\Psi|_{\mathcal{C}_b}$, $Y_1|_{\mathcal{C}_b}$, $Y_2|_{\mathcal{C}_b}$ and $F \circ \Psi|_{\mathcal{C}_b}$ are Lipschitz continuous. In particular, $\Psi|_{\mathcal{C}_b} : \mathcal{C}_b \to C_b$ is a Lipschitz homeomorphism.*

Proof. Let $u, v \in \mathcal{C}_b$, and let $r = R(u)$, $s = R(v)$, $\psi = \Psi(u)$, $\sigma = \Psi(v)$. In particular,

$$\frac{4r}{b} = \int_0^{2\pi} u(\varphi)\mathrm{e}^{r\cos 2\varphi}\cos 2\varphi\,\mathrm{d}\varphi, \qquad \frac{4s}{b} = \int_0^{2\pi} v(\varphi)\mathrm{e}^{s\cos 2\varphi}\cos 2\varphi\,\mathrm{d}\varphi.$$

Assume without loss of generality that $0 < s < r \leqslant b/4$. We first consider the case when $b > 8$. Then

$$\begin{aligned}
\frac{4(r-s)}{b} &= \int_0^{2\pi} (u-v)\mathrm{e}^{r\cos 2\varphi}\cos 2\varphi\,\mathrm{d}\varphi + \int_0^{2\pi} v(\mathrm{e}^{r\cos 2\varphi} - \mathrm{e}^{s\cos 2\varphi})\cos 2\varphi\,\mathrm{d}\varphi \\
&= \int_0^{2\pi} (u-v)\mathrm{e}^{r\cos 2\varphi}\cos 2\varphi\,\mathrm{d}\varphi + \int_0^{2\pi} \sigma(\mathrm{e}^{(r-s)\cos 2\varphi} - 1)\cos 2\varphi\,\mathrm{d}\varphi \\
&= \int_0^{2\pi} (u-v)\mathrm{e}^{r\cos 2\varphi}\cos 2\varphi\,\mathrm{d}\varphi + h'(r-s) + \int_0^{2\pi} (\sigma - \bar{\psi})(\mathrm{e}^{(r-s)\cos 2\varphi} - 1)\cos 2\varphi\,\mathrm{d}\varphi \\
&\geqslant \int_0^{2\pi} (u-v)\mathrm{e}^{r\cos 2\varphi}\cos 2\varphi\,\mathrm{d}\varphi + h'(r-s) - 2\pi\|\sigma - \bar{\psi}\|_\infty h'(r-s) \\
&\geqslant \int_0^{2\pi} (u-v)\mathrm{e}^{r\cos 2\varphi}\cos 2\varphi\,\mathrm{d}\varphi + \left(1 - \frac{8\pi M_0 s}{\sqrt{b}}\right) h'(r-s).
\end{aligned}$$

Note that for the last estimate, we used the fact that u and v are in the absorbing cone $\mathcal{C}_b$. Since $h'''(x) > 0$ for $x > 0$ and since $h''(0) = 1/2$, we have

$$\begin{aligned}
\left(\frac{1}{2} - \frac{4}{b} - \frac{4\pi M_0 s}{\sqrt{b}}\right)(r-s) &\leqslant \int_0^{2\pi} (v-u)\mathrm{e}^{r\cos 2\varphi}\cos 2\varphi\,\mathrm{d}\varphi \\
&\leqslant C(b)\|u-v\|_2,
\end{aligned}$$

and the Lipschitz continuity follows when $0 < s \leqslant s_0$, for $s_0 = s_0(b)$ small enough. Inequality (3.14) implies that for $s > s_0$

$$r - s \leqslant \frac{1}{\hat{v}'(s)} \int_0^{2\pi} (v(\varphi) - u(\varphi))\mathrm{e}^{r\cos 2\varphi}\,\mathrm{d}\varphi \leqslant \frac{bC(b)}{4s_0}\|u-v\|_2,$$

and this completes the proof in the case $b > 8$.

Now consider the case $0 < b < 8$. Similarly as before we have

$$\frac{4(r-s)}{b} \leqslant \int_0^{2\pi} (u-v)\mathrm{e}^{r\cos 2\varphi}\cos 2\varphi\,\mathrm{d}\varphi + \left(1 + \frac{8\pi M_0 s}{\sqrt{b}}\right) h'(r-s).$$

We again exploit the fact that h'' is increasing on and that $h''(0) = 1/2$. In this case, for $0 < \epsilon < 4/b - 1/2$ there exists $\delta > 0$, so that for $r - s < \delta$

$$\begin{aligned}
\left(\frac{4}{b} - \frac{1}{2} - \frac{4\pi M_0 s}{\sqrt{b}} - \epsilon\right)(r-s) &\leqslant \int_0^{2\pi} (u-v)\mathrm{e}^{r\cos 2\varphi}\cos 2\varphi\,\mathrm{d}\varphi \\
&\leqslant C_b\|u-v\|_2,
\end{aligned}$$

and the Lipschitz continuity follows when $0 < s \leqslant s_0$, for $s_0 = s_0(b)$ small enough, and when $r - s < \delta$. In the case that $r - s \geqslant \delta$, we define the sets $M_s = R^{-1}[0, s]$ and $N_r = R^{-1}[r, \infty)$. These are disjoint closed sets, so $\mathrm{dist}(M_s, N_r) > 0$, and since $r, s \in [0, b/4]$,

$\min_{r-s\geqslant\delta} \operatorname{dist}(M_s, N_r) = C_\delta > 0$. In this case we have

$$r - s \leqslant \frac{b}{C_\delta}\|u - v\|_2.$$

The Lipschitz continuity for all $s < r$ follows as before from (3.14). □

We now expand the Lipschitz continuity to a larger set whose interior in L^2 contains $U_b^+ := \Psi^{-1}(\psi_b^+)$ corresponding to the fixed intensity $b \neq 8$.

Lemma 3. *Let $0 < b \neq 8$. Let $b \in (b_1, b_2)$ and $0 < r_1 < r_b^+$. Let $\mathcal{B}' = \mathcal{C}_b \cup \mathcal{B}$, where $\mathcal{B} = B^{-1}(b_1, b_2) \cap R^{-1}(r_1, b_2/4)$. Then the functions $R|_{\mathcal{B}'}$, $\Psi|_{\mathcal{B}'}$, $Y_1|_{\mathcal{B}'}$, $Y_2|_{\mathcal{B}'}$, $B|_{\mathcal{B}'}$ and $F \circ \Psi|_{\mathcal{B}'}$ are Lipschitz continuous.*

Proof. Let us divide $\mathcal{B}'$ into three regions: $P = \mathcal{C}_b \cap R^{-1}(0, r_1/2]$, $Q = \mathcal{C}_b \cap R^{-1}[r_1/2, r_1]$, and $S = B^{-1}(b_1, b_2) \cap R^{-1}(r_1, b_2/4)$. The previous lemma and inequality (3.14) imply the Lipschitz continuity on each one of these regions separately. It remains to prove that it still holds on the union of any of the pairs of these regions. This obviously holds on $P \cup Q \subset \mathcal{C}_b$, and on $Q \cup S$ due to (3.14). For the set $P \cup S$ we use the fact that $P \subset M_{r_1/2}$ and $S \subset N_{r_1}$, where M and N are the sets defined in the proof of the previous lemma, and the conclusion follows similarly. This concludes the proof. □

We can now define the prepared equation. Note that $\mathcal{C}_b \subset B_{\rho/2} = \{u \in L^2[0, 2\pi] : u \text{ even}, \|u\|_2 \leqslant \rho/2\}$, where $\rho = 2e^{b/4}(M_0\sqrt{2\pi b} + 1/\sqrt{2\pi})$. We define

$$N_P(u) = \begin{cases} F[\Psi(u)]u, & \text{if } u \in \mathcal{B}' \cap B_{\rho/2}, \\ 0, & \text{if } u \in B_\rho^c. \end{cases}$$

This is clearly a Lipschitz function on $(\mathcal{B} \cap B_{\rho/2}) \cup B_\rho^c$. It is a well-known fact from analysis that there exists a Lipschitz continuous extension of N_P defined on the entire $\mathcal{H} = \{u \in L^2[0, 2\pi] : u \text{ even}\}$. We obtain the prepared equation

$$\partial_t u + Au = N_P(u),$$

where $A = -\partial_{\varphi\varphi}$.

3.3. Main theorem

We are now able to prove the existence of the inertial manifold of the Smoluchowski equation.

Theorem 3. *Let $b \in \mathbb{R}^+\backslash\{8\}$. The Smoluchowski equation on the unit circle with the Maier–Saupe potential possesses an asymptotically complete inertial manifold $\mathcal{M}_b$.*

Proof. The positivity of A and the Lipschitz continuity of N_P ensure that the prepared equation generates a strongly continuous semigroup $S_P(t)$. The fact that N_P vanishes outside of B_ρ suffices to prove that the prepared equation is dissipative and that it possesses a finite-dimensional global attractor $\mathcal{A}_P$. Also, it can easily be observed that $u_b^+ := \Psi^{-1}(\psi_b^+) \in \mathcal{A}_P$. The complete set of eigenfunctions for the linear operator A is given by $w_k(\varphi) = \cos k\varphi$, $k = 0, 1, \ldots$, with eigenvalues $\lambda_k = k^2$, $k = 0, 1, \ldots$. If C_1 is a Lipschitz constant for N_P, there exists $n \in \mathbb{N}$ such that $\lambda_{n+1} - \lambda_n = 2n + 1 > 4C_1$, and the spectral gap condition is satisfied. Theorem 2 applies, and we infer the existence of an asymptotically complete inertial manifold $\mathcal{M}_P \supset \mathcal{A}_P$ for the prepared equation given as a graph of a Lipschitz function Φ_P:

$$\mathcal{M}_P = \mathcal{G}[\Phi_P] = \{p + \Phi_P(p) : p \in P_n\mathcal{H}\}.$$

We now define $\mathcal{M}_b = C_b \cap \Psi(\mathcal{M}_P)$. Since $\Psi|_{\mathcal{C}_b} : \mathcal{C}_b \to C_b$ is a Lipschitz homeomorphism, it is immediate that $\mathcal{M}_b$ is a finite-dimensional Lipschitz manifold. It is positively invariant under $S(t)$, since both sets C_b and $\Psi(\mathcal{M}_P)$ are positively invariant. It remains to prove that $\mathcal{M}_b$ is exponentially attracting and asymptotically complete. Let $\psi_0 \in H$ and $\psi(t) = S(t)\psi_0$. Let $u(t) = \Psi^{-1}(\psi(t))$, $t \geqslant 0$. Since $\psi(t) \to \psi_b^+$ as $t \to \infty$, $u(t) \to u_b^+$ as $t \to \infty$. On the other hand, since $\mathcal{M}_P$ is exponentially attracting and asymptotically complete, there exists $v_0 \in \mathcal{M}_P$ so that for $v_P(t) = S_P(t)v_0$ we have $\|u(t) - v_P(t)\|_2 \to 0$, as $t \to \infty$, exponentially. Thus, $v_P(t) \to u_b^+$ as $t \to \infty$ as well. Since $u_b^+ \in \mathcal{B} \cap B_{\rho/2}$, there exists $T > 0$ so that $v_P(t) \in \mathcal{B} \cap B_{\rho/2}$ for $t \geqslant T$. However, since $N_P|_{\mathcal{B}\cap B_{\rho/2}} = N|_{\mathcal{B}\cap B_{\rho/2}}$, $B(v_P(t)) = B(u_b^+) = b$ for $t \geqslant T$. Therefore, $\sigma(t) := \Psi(v_P(t)) \in \Psi(\mathcal{M}_P)$, $t \geqslant T$ is a solution of (2.1). For some $T' \geqslant T$ we have $\sigma(t) \in C_b$, $t \geqslant T'$, and therefore $\sigma(t) \in \mathcal{M}$, $t \geqslant T'$. Since $\Psi|_{\mathcal{C}_b}$ is Lipschitz continuous, $\|\psi(t) - \sigma(t)\|_2 \to 0$, as $t \to \infty$, exponentially. This concludes the proof. □

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