Inertial Manifolds for a Smoluchowski Equation on the Unit Sphere

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Abstract

The existence of inertial manifolds for a Smoluchowski equation – a nonlinear Fokker-Planck equation on the unit sphere which arises in modeling of colloidal suspensions – is investigated. A nonlinear and nonlocal transformation is used to eliminate the gradient from the nonlinear term.

Keywords: Smoluchowski equation; global attractor; inertial manifolds
Mathematics subject classification: 35Kxx, 70Kxx

1 Introduction

Although intrinsically infinite-dimensional, many dissipative parabolic systems exhibit long-term dynamics with properties typical of finite-dimensional dynamical systems. The global attractor, often considered the central object in the study of long-term behavior of dynamical systems, appears to be inadequate in capturing this finite-dimensionality, even when its Hausdorff dimension is finite. This is mainly due to two facts. Firstly, the global attractor can be a very complicated set, not necessarily a manifold; the dynamics on it may not be reducible to a finite system of ODEs. Secondly, although all solutions approach this set, they do so at arbitrary rates, algebraic or

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exponential, and thus dynamics outside the attractor is not tracked very well on the attractor itself. When they exist, inertial manifolds emerge as most adequate objects to capture the finite-dimensionality of a dissipative parabolic PDE. Introduced by Foias at al. in [15], they are defined to remedy the shortcomings of the global attractor just described: they should be finite dimensional positive-invariant Lipschitz manifolds which attract all solutions exponentially, and on which the solutions of the underlying PDE are recoverable from solutions of a system of ODEs, termed inertial form. Their existence does not only have a theoretical value, but a practical one as well: using a system of ODEs instead of a system of PDEs facilitates computations and numerical analysis.

One of the most notable examples of parabolic PDEs which possess inertial manifolds is the Kuramoto-Sivashinsky equation [14]. However, the existence of inertial manifolds remains to date unattainable for the vast majority of physically relevant dissipative PDEs; chief amongst them is the 2D Navier-Stokes system, which possesses a finite-dimensional attractor, for which, however, the existence of inertial manifolds is still open. The main reason for this lies in the fact that most methods (except in some very special cases) require the system at hand to satisfy a very restrictive spectral-gap condition; this condition is especially restrictive when first-order derivatives are present in the nonlinear term, as is the case for the Navier-Stokes equations.

The Smoluchowski equation describes the temporal evolution of the probability distribution function ψ for directions of rod-like particles in a suspension. The equation has the form of a Fokker-Planck equation

$$\partial_t \psi = \Delta \psi + \nabla \cdot (\psi \nabla V).$$

It is, however, phrased on the unit sphere, and the gradient, the divergence and the Laplacian are correspondingly modified. The unknown function V stands for a mean-field potential resulting from the excluded volume effects due to steric forces between molecules. Unlike the Fokker-Planck equation, the equation is quadratically nonlinear due to the dependence of V on the probability distribution ψ, and it is nonlocal, since this dependence is nonlocal. In that respect, the equation is similar to the Keller-Segel model of chemotaxis. In this paper, we shall use a particular type of mean-field potential due to Maier and Saupe [22], which can be thought of as the projection of the probability distribution function on the second eigenspace of the Laplace-Beltrami operator multiplied by a constant.

The Smoluchowski equation was first proposed in the works of Doi [10] and Hess [17] as a dynamical model for nematic liquid crystalline polymers. In this paper, we study a version of the equation in which the interaction with the ambient flow is neglected in which case the equation is a gradient system with the free energy as the Lyapunov functional. It is dissipative in a Gevrey class of analytic functions, and it possesses a finite-dimensional global attractor consisting of the steady-states and their unstable manifolds. Historically, the Smoluchowski equation was preceded by a variational model for colloidal suspensions due to Onsager [23]. Onsager calculated the free energy functional and derived the Euler-Lagrange equation for the steady-states. The mean-field potential used in his work was different, and the Maier-Saupe potential is a truncation of this potential. However, it is widely accepted that the Maier-Saupe potential affords sufficient degrees of freedom to capture the dynamics of the micro-micro interaction. In a recent development, the bifurcation diagram for the Onsager equation (and therefore also Smoluchowski equation) with the Maier-Saupe potential was confirmed rigorously (see [5], [6], [8], [13], [18], [19]). The equation undergoes two bifurcations. At a lower potential intensity, the equation undergoes a saddle-node bifurcation, in which a prolate nematic branch of steady-states (probability distribution concentrates to one direction) and an oblate nematic branch of steady-states (probability distribution concentrates uniformly to the equator) emerge. At a higher potential intensity, the equation undergoes a transcritical bifurcation: the oblate branch intersects with the isotropic state, and there is a transfer of stability.

All these facts point to the finite-dimensionality of the dynamics. However, just like for the 2D Navier-Stokes system, the Smoluchowski equation has a gradient in the nonlinear term, and it does not satisfy the spectral-gap condition. This difficulty is circumvented in this paper by a nonlinear nonlocal transformation which eliminates the gradient from the nonlinearity. The dynamics of the Smoluchowski equation becomes much more complex when we allow for interaction with the ambient flow. Even a passive interaction with a shear flow leads to complicated and peculiar dynamical behavior. This is due to the fact that the fluid introduces a non-variational element to the dynamics, even if the nonlinearity remains unchanged. In addition to flow-aligning (steady-states), different time-periodic solution regimes and chaos were confirmed numerically. The method developed here can be modified to prove the existence of inertial manifolds in this dynamically more interesting case, as well as for other equations of a similar type.

2 Preliminaries

2.1 Smoluchowski equation on a unit sphere

We consider the Smolukowski equation on the unit sphere ($\mathbf{m}\in S^2$)

$$\partial_t\psi = \Delta_{\mathbf{m}}\psi + \nabla_{\mathbf{m}}\cdot(\psi\nabla_{\mathbf{m}}V), \tag{2.1}$$

where $\nabla_{\mathbf{m}} = \mathbf{m}\times\partial_{\mathbf{m}}$ stands for the gradient operator on the unit sphere, and $\Delta_{\mathbf{m}} = \nabla^2_{\mathbf{m}}$ stands for the Laplace-Beltrami operator. We also write the equation in the functional form

$$\partial_t\psi + A\psi = B(\psi, V), \tag{2.2}$$

where the linear operator A and the bilinear operator B are defined as $A = -\Delta_{\mathbf{m}}$ and $B(\psi,\xi) = \nabla_{\mathbf{m}}\cdot(\psi\nabla_{\mathbf{m}}\xi)$, respectively. The unknown ψ is interpreted as the probability distribution function for the orientations of rigid rod-like molecules in a suspension. The simplest quantity representing its anisotropy is the orientational order-parameter tensor which is calculated as the traceless equivalent of the second moment tensor:

$$\mathcal{S}[\psi(t)] = <\mathbf{mm} - \mathrm{I}/3>_{\psi(t)} = \int_{S^2}[\mathbf{mm} - \mathrm{I}/3]\,\psi(\mathbf{m},t)\;d\mathbf{m}.$$

The scaler order parameter

$$S[\psi] = (3/2\mathcal{S}[\psi]:\mathcal{S}[\psi])^{\frac{1}{2}} \in [0,1]$$

represents the degree of molecular alignment. For the isotropic phase $\bar{\psi} = 1/4\pi$, $S[\bar{\psi}] = 0$, and for the perfect alignment $S[\psi] = 1$. The unknown V is a mean-fild intermolecular interaction potential resulting from the excluded volume effects due to steric forces between molecules. In this paper, we utilize the Maier-Saupe potential given by

$$V(\mathbf{m},t) = -b(\mathbf{mm} - \mathrm{I}/3):\mathcal{S}[\psi(t)], \tag{2.3}$$

where the parameter $b > 0$ represents the potential intensity. Due to the nonlocal dependence of V on the probability distribution function ψ, the Smoluchowski equation is nonlinear (quadratic) and nonlocal. Note that

$$A(\mathbf{mm} - \mathrm{I}/3) = \lambda_2(\mathbf{mm} - \mathrm{I}/3),$$

and therefore also $AV = \lambda_2 V$, where $\lambda_2 = 6$ is the second smallest positive eigenvalue of A. More specifically,

$$V(\mathbf{m},t) = -b\frac{8\pi}{15}P_2\psi,$$

where P_2 is the projection on the spectral eigenspace of the operator A corresponding to the eigenvalue $\lambda_2 = 6$.

Regarding the existence, uniqueness and regularity of solutions of (2.1) with (2.3), it is easy to prove the following theorem (see [5], [6]).

Theorem 1 *Let $\psi_0 > 0$ be a continuous function on S^1 such that $\int_{S^2}\psi = 1$. A unique smooth solution $\psi(t) = S(t)\psi_0$ of (2.1) and (2.3) with initial datum $\psi(0) = \psi_0$ exists for all nonnegative times and remains positive and normalized*

$$\int_{S^2} \psi(\mathbf{m},t)\ d\mathbf{m} = 1.$$

The Smoluchowski equation preserves certain symmetries. Symmetry with respect to the origin – reflecting the fact that that we do not distinguish between orientations $\mathbf{m}$ and $-\mathbf{m}$ – is preserved. Also, symmetry with respect to any plane passing through the origin is preserved. This allows us to choose a coordinate system, so that the symmetry with respect to any of the coordinate planes is preserved. We will refer to functions with these symmetries simply as 'symmetric'. We express ψ in terms of the local coordinates: $\psi(\theta,\varphi) = \psi(\mathbf{m}(\theta,\varphi))$, where $m_1(\theta,\varphi) = \sin\theta\cos\varphi$, $m_2(\theta,\varphi) = \sin\theta\sin\varphi$, and $m_3(\theta,\varphi) = \cos\theta$. Then, in terms of the local coordinates

$$\nabla_{\mathbf{m}}\psi = \left(\partial_\theta\psi, \frac{1}{\sin\theta}\partial_\varphi\psi\right)$$

and

$$\Delta_{\mathbf{m}}\psi = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\ \partial_\theta\psi) + \frac{1}{\sin^2\theta}\partial_\varphi^2\psi.$$

Due to the choice of the coordinate system, the order parameter tensor is a diagonal trace-free matrix

$$\mathcal{S}[\psi] = \begin{bmatrix} \langle m_1^2 - 1/3\rangle_\psi & 0 & 0 \\ 0 & \langle m_2^2 - 1/3\rangle_\psi & 0 \\ 0 & 0 & \langle m_3^2 - 1/3\rangle_\psi, \end{bmatrix}.$$

This enables to rewrite the potential in the following way:

$$
\begin{aligned}
V(\mathbf{m},t) &= -b[\left\langle m_1^2-1/3\right\rangle_\psi (m_1^2-1/3)+\left\langle m_2^2-1/3\right\rangle_\psi (m_2^2-1/3)+\left\langle m_3^2-1/3\right\rangle_\psi (m_3^2-1/3)]\\
&= -\frac{b}{2}[\left\langle m_1^2-m_2^2\right\rangle_\psi (m_1^2-m_2^2)+3\left\langle m_3^2-1/3\right\rangle_\psi (m_3^2-1/3)\\
&= -\frac{b}{2}[\langle w_1\rangle_\psi w_1+\langle w_2\rangle_\psi w_2]\\
&= \frac{b}{2}\langle \mathbf{w}\rangle_\psi \cdot \mathbf{w},
\end{aligned}
$$

where $w_1=m_1^2-m_2^2=\sin^2\theta\cos 2\varphi$, $w_2=\sqrt{3}(m_3^2-1/3)=\sqrt{3}(\cos^2\theta-1/3)$, and $\mathbf{w}=(w_1,w_2)$. Notice that $\|w_1\|_2^2=\|w_2\|_2^2=\frac{16\pi}{15}$. Multiplying (2.1) by $m_i^2-1/3$ and integrating by parts yields the equation for temporal evolution of the order parameter tensor

$$
\partial_t \mathcal{S}_{ii}[\psi]+6\mathcal{S}_{ii}[\psi]=4b\left[\mathcal{S}_{ii}[\psi]\left\langle m_i^2\right\rangle_\psi-\sum_{k=1}^{3}\mathcal{S}_{kk}[\psi]\left\langle m_i^2 m_k^2\right\rangle_\psi\right]. \tag{2.4}
$$

In particular, this equation yields the equation for the temporal evolution of the mean-field potential

$$
\partial_t V+6V=G[\psi], \tag{2.5}
$$

where $G[\psi]:S^2\to \mathbb{R}$ depends Lipschitz-continuously on second and fourth moment tensors of ψ only.

2.2 Dissipativity and the global attractor

Here we review some of basic properties of the spherical harmonics. Let P_k denote the Legendre polynomial of degree k. For $k=0,1,2,\ldots$ and $j=0,\pm1,\pm2,\ldots,\pm k$ we define

$$
Y_k^j(\theta,\varphi)=C_k^j e^{ij\varphi}P_k^j(\cos\theta),
$$

where $C_k^j=\left[\frac{2k+1}{4\pi}\frac{(k-|j|)!}{(k+|j|)!}\right]^{1/2}$, $P_k^j(x)=(1-x^2)^{j/2}\frac{d^jP_k}{dx^j}(x)$, $j=0,1,2,\ldots,k$, and $P_k^j=P_k^{-j}$, $j=-1,-2,\ldots,-k$. Each Y_k^j is an eigenvector of $A=-\Delta_{\mathbf{m}}$ corresponding to the eigenvalue $\lambda_k=k^2+k$: $AY_k^j=\lambda_k Y_k^j$. Moreover, the set $\{Y_k^j: k=0,1,2,\ldots; j=0,\pm1,\pm2,\ldots,\pm k\}$ forms an orthonormal basis in $L^2(S^2)$; in particular, for each $\psi\in L^2(S^2)$, there is a representation

$$
\psi=\sum_{k=0}^{\infty}\sum_{j=-k}^{k}\psi_k^j Y_k^j,
$$

where

$$\psi_k^j = \int_{S^2} \psi Y_k^{-j}\ d\mathbf{m}.$$

We define

$$P_k\psi = \sum_{j=-k}^{k} \psi_k^j Y_k^j.$$

Let $H = \{\psi \in L^2(S^2) : \psi$ normalized and symmetric$\}$. For $\psi \in H$, the symmetry implies $\psi_k^{-j} = \bar{\psi}_k^j = \psi_k^j$, $\psi_{2k+1}^j = 0$, and the normalization yields

$$\psi_0^0 = \frac{1}{\sqrt{4\pi}}$$

and

$$|\psi_k^j| \le \int_{S^2} \psi |Y_k^{-j}|\ d\mathbf{m} \le \sqrt{\frac{2k+1}{4\pi}}.$$

We will make use of the fact that $\|P_2\psi\|_\infty \le \frac{5}{4\pi}$.

For the scalar order parameter we now have

$$S[\psi]^2(t) = -\frac{3}{2b}\int_{S^2} V(\mathbf{m},t)\psi(\mathbf{m},t)\ d\mathbf{m} = \frac{4\pi}{5}\|P_2\psi\|_2^2 = \frac{3}{4}(\langle w_1\rangle_\psi^2 + \langle w_2\rangle_\psi^2).$$

After multiplying (2.1) by $-V(\mathbf{m},t)$ and integrating by parts, we obtain

$$\frac{d}{2dt}S[\psi]^2 + 6S[\psi]^2 = \frac{3}{2b}\int_{S^2} |\nabla_{\mathbf{m}} V|^2 \psi\ d\mathbf{m}. \tag{2.6}$$

On the other hand, by multiplying (2.4) by $\mathcal{S}_{ii}[\psi]$ and summing, one obtains

$$\frac{d}{2dt}S[\psi]^2 + \left(6 - \frac{4b}{5}\right) S[\psi]^2 = b\sum_{i=1}^{3}\sum_{k=1}^{3} \mathcal{S}_{ii}[\psi]\mathcal{S}_{kk}[\psi]\mathcal{T}_{ik}[\psi],$$

where $\mathcal{T}_{ij}[\psi] = < T_{ij}(\mathbf{m}) >_\psi$, and $(T_{ij}(\mathbf{m})) \in P_2(L^2(S^2)) \oplus P_4(L^2(S^2))$ is symmetric. If $\psi(t) \to \bar{\psi} = 1/4\pi$ as $t \to \infty$, then $\mathcal{T}[\psi(t)] \to \mathcal{O}$ as $t \to \infty$, so if $b < 15/2$, then $S[\psi(t)] \to 0$ exponentially as $t \to \infty$, and if $b > 15/2$, then $S[\psi(t)] \equiv 0$ or $S[\psi(t)] \to \infty$ as $t \to \infty$, which cannot be the case. Note that if $S[\psi(0)] \equiv 0$, then $S[\psi(t)] \equiv 0$ ($V \equiv 0$), and the Smoluchowski equation reduces to the heat equation in this case. We again have $\psi(t) \to \bar{\psi}$ as $t \to 0$, exponentially.

In the following we define Gevrey classes of functions. For $a>1$, we define on the set of eigenvalues of A the function $f(\lambda_k)=a^{2k}$. We define a spectral operator $\mathcal{F}=f(A)$ by

$$\mathcal{F}\psi=\sum_{k=2}^{\infty}f(\lambda_k)P_k\psi$$

Let $(\cdot,\cdot)_{\mathcal{F}}=(\cdot,\mathcal{F}(\cdot))_{L^2(S^1)}$, $\|\cdot\|_{\mathcal{F}}=(\cdot,\cdot)_{\mathcal{F}}^{1/2}$, and $H_{\mathcal{F}}=\{\psi\in H:\|\psi\|_{\mathcal{F}}<\infty\}$. From [7], we adopt a Lemma concerning the nonlinear term in the equation:

Lemma 1 *For $\psi,\chi,\xi\in D(A)$*

$$(B(\psi,\xi),\chi)_{L^2(S^1)}=\frac{1}{2}\int_{S^2}[\xi\chi A\psi-\psi\xi A\chi-\psi\chi A\xi]\ d\mathbf{m}. \tag{2.7}$$

With a minor modification, we also adopt the following

Lemma 2 *There exist an absolute constant $C_1>0$ and a constant $C_2=C_2(b)>0$ depending on b only, such that, if $1\le a^4\le 1+(4C_1b)^{-1}$, then for any $\psi\in H\cap D(A\mathcal{F}(A))$ and V computed through (2.3) the inequality*

$$|(B(\psi,V),\psi)_{\mathcal{F}}|\le C_2^2+\frac{1}{2}((A-\lambda_2)\psi,\psi)_{\mathcal{F}}$$

holds.

This enables us to prove the following theorem on the existence of absorbing cones in Gevrey classes:

Theorem 2 *Let $\psi_0\in H_{\mathcal{F}}$ and $\psi(t)$ the unique solution of (2.2) corresponding to that initial datum, and $S[\psi(t)]$ the scalar order parameter. Let the number a be such that $1<a^4\le 1+(4C_1b)^{-1}$. Then*

$$\frac{\|\psi(t)\|_{\mathcal{F}}^2}{S[\psi(t)]^2}\le 2C_2^2+e^{-t}\frac{\|\psi_0\|_{\mathcal{F}}^2}{S[\psi(0)]^2},\quad t\ge 0. \tag{2.8}$$

In particular, the cone $\{\psi\in H_{\mathcal{F}}:\|\psi\|_{\mathcal{F}}\le 2C_2S[\psi]\}$ is absorbing and invariant.

Proof : Multiplying the equation (2.2) by $\mathcal{F}(t)\psi$ and integrating over S^2 one obtains

$$\frac{d}{2dt}(\psi,\mathcal{F}\psi)+(A\psi,\mathcal{F}\psi)=(B(\psi,V),\mathcal{F}\psi).$$

Let $\tilde{\psi} := \psi/S[\psi]$ and $\tilde{V} := V/S[\psi]$. In view of (2.6)

$$\frac{d}{2dt}(\tilde{\psi},\tilde{\psi})_{\mathcal{F}} + ((A-\lambda_2)\tilde{\psi},\tilde{\psi})_{\mathcal{F}} \leq (B(\tilde{\psi}/,\tilde{V}),\mathcal{F}\tilde{\psi})S[\psi],$$

and therefore in view of Lemma 2

$$\frac{d}{2dt}(\tilde{\psi},\tilde{\psi})_{\mathcal{F}} + \frac{1}{2}((A-\lambda_2)\tilde{\psi},\tilde{\psi})_{\mathcal{F}} \leq C_2^2,$$

and so

$$\frac{d}{dt}(\tilde{\psi},\tilde{\psi})_{\mathcal{F}} + (\tilde{\psi},\tilde{\psi})_{\mathcal{F}} \leq 2C_2^2,$$

and the statement follows. □

The dissipativity in Gevrey classes implies the dissipativity of ψ in $L^\infty(S^2)$ and therefore also in $L^2(S^2)$. By $B_{\rho(b)} = \{\psi \in H : \|\psi\|_2 < \rho(b)\}$ let us denote an absorbing invariant ball for (3.11). We chose $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ to be continuous and increasing. Another consequence of the theorem is the existence of a finite dimensional global attractor $\mathcal{A}$ whose structure is somewhat simple due to the fact that the equation (2.1) is a gradient system. For dissipative gradient systems, the dynamical behavior is characterized by a global attractor which is formed by the steady-states and their unstable manifolds, and all solutions approach a steady-state as $t \to \infty$. Denoting $u = \psi \exp(V/2)$, the Lyapunov functional for the Smoluchowski equation is given by

$$\mathcal{F}(t) = \int_{S^2} \psi(\mathbf{m},t) \log u(\mathbf{m},t)\ d\mathbf{m},$$

which satisfies the equation

$$\frac{d\mathcal{F}}{dt} = -\int_{S^2} |\nabla_{\mathbf{m}}(V + \log\psi)|^2 \psi\ d\mathbf{m}.$$

The right-hand side of the energy dissipation equation yields the equation for the steady-states for the Smoluchowski equation which coincides with the Onsager equation which preceded the Hess-Doi theory. The equation reads

$$V + \log\psi = \text{const.},$$

or, by denoting $\mathbf{r} = (b/4) < \mathbf{w} >_\psi$ and solving for ψ,

$$\psi(\mathbf{m}) = \frac{\exp(2\mathbf{r}\cdot\mathbf{w}(\mathbf{m}))}{\int_{S^2} \exp(2\mathbf{r}\cdot\mathbf{w}(\mathbf{m}))\ d\mathbf{m}}. \tag{2.9}$$

This transcendental matrix equation is now well understood (see [5], [6], [8], [13], [18])). The isotropic steady state $\bar{\psi} = 1/4\pi$ persists for all potential intensities $b > 0$. All anisotropic steady-states are axisymmetric with the coordinate axes as axes of symmetry. There are two critical intensities at which bifurcations occur: a saddle-node bifurcation at $b = b^* = 6.7314863965$ in which three oblate and three prolate anisotropic steady-states emerge (one for each coordinate axis as the axis of symmetry), and a transcritical bifurcation at $b = b^{**} = 15/2$ in which the oblate branch intersects with the isotropic steady-state. If $b < b^{**}$, then $\psi = \bar{\psi}$ is asymptotically stable. In view of the fact that $\psi(t) \to \bar{\psi}$ as $t \to \infty$ implies $S[(\psi(t)] \to 0$ as $t \to \infty$, exponentially, and in view of (2.8), $\psi(t) \to \bar{\psi}$ as $t \to \infty$, exponentially. If $b > b^{**}$, $\bar{\psi}$ becomes a saddle with the set $\{\psi : \ S[\psi] = 0\}$ as the basin of attraction. All anisotropic steady-states have the form (2.9) for some $\mathbf{r} \in \mathbb{R}^2$. By $r^*(b)$ we denote the least $|\mathbf{r}|$.

2.3 Inertial Manifolds

In this section, we recall the definition of inertial manifolds and a theorem on their existence. Consider an evolution equation on a Hilbert space H endowed with the inner product $(\cdot,\cdot)$, and the norm $|\cdot|$ of the form

$$\partial_t u + Au = N(u), \tag{2.10}$$

where A is a positive self-adjoint linear operator with compact inverse, and $N : H \to H$ is a locally Lipschitz function. Recall that, since A^{-1} is compact, there exists a complete set of eigenfunctions w_k for A

$$Aw_k = \lambda_k w_k.$$

We arrange the eigenvalues in a nondecreasing sequence $\lambda_k \le \lambda_{k+1}$, $k = 1, 2, \ldots$ It is a well-know fact that $\lambda_k \to \infty$ as $k \to \infty$. We also define the projection operators

$$P_n u = \sum_{k=1}^{n} (u, w_k) w_k$$

and $Q_n = I - P_n$, and the the cone-like sets

$$\mathcal{C}_l^n = \{w \in H : |Q_n w| \le l |P_n w|\}.$$

Definition 1 *An inertial manifold $\mathcal{M}$ is a finite-dimensional Lipschitz manifold which is positively invariant*

$$S(t)\mathcal{M} \subset \mathcal{M},\ t \geq 0,$$

and exponentially attracts all orbits of the flow uniformly on any bounded set $U \subset H$ of initial data

$$dist(S(t)u_0, \mathcal{M}) \leq C_U e^{-\mu t},\ u_0 \in U,\ t \geq 0.$$

The inertial manifold is said to be asymptotically complete, if, for any solution $u(t)$, there exists $v_0 \in \mathcal{M}$ such that

$$|u(t) - S(t)v_0| \to 0, \quad t \to \infty,$$

exponentially.

There are several methods for proving the existence of inertial manifolds. The vast majority of them require some kind of Lipschitz continuity of the nonlinearity N and make use of a very restrictive spectral gap property of the linear operator A which is usually the Laplacian. These two conditions yield the strong squeezing property, which, in turn, yields the existence of an inertial manifold. The inertial manifold is obtained as a graph of a Lipschitz mapping.

Theorem 3 *Suppose that the nonlinearity in (2.10) satisfies the following three conditions*

- *It has compact support in H, i.e. $supp(N) \subset B_\rho = \{u \in H : |u| \leq \rho\}$*
- *It is bounded, i.e. $|N(u)| \leq C_0$ for $u \in H$*
- *It is globally Lipschitz continuous, i.e. $|N(u_1) - N(u_2)| \leq C|u_1 - u_2|$ for $u_1, u_2 \in H$.*

Suppose that the eigenvalues of A satisfy the spectral gap condition, i.e.

$$\lambda_{n+1} - \lambda_n > 4C,$$

for some $n \in \mathbb{N}$. Then the strong squeezing property holds, i.e.

- *If $u_1(0) - u_2(0) \in \mathcal{C}_l^n$, then $u_1(t) - u_2(t) \in \mathcal{C}_l^n$ for $t \geq 0$.*

- *If* $u_1(t_0)-u_2(t_0)\notin\mathcal{C}_l^n$, *for some* $t_0\geq 0$, *then* $|Q_n(u_1(t)-u_2(t))|\leq |Q_n(u_1(0)-u_2(0))|e^{-\mu t}$ *for some* $\mu>0$ *and for* $0\leq t\leq t_0$.

The strong squeezing property implies the existence of an asymptotically complete inertial manifold which is the graph of a Lipschitz function $\Phi:P_nH\to Q_nH$, *i.e.*

$$\mathcal{M}=\mathcal{G}[\Phi]=\{p+\Phi(p):p\in P_nH\}$$

with

$$|\Phi(p_1)-\Phi(p_2)|\leq l|p_1-p_2|.$$

Restricting (2.10) to $\mathcal{M}$ *yields the ordinary differential equation for* $p=P_nu$

$$\frac{dp}{dt}+Ap=P_nN(p+\Phi(p))$$

termed the inertial form.

Different proofs are available in the literature (e.g. [3], [24], [25]). The above result is not the strongest possible. It is possible to ease the Lipschitz condition to allow for nonlinearities that contain first-order derivatives of u, resulting, however, in an even more restrictive spectral gap condition, which is not satisfied by the Smoluchowski equation. The main idea of this paper is to eliminate the gradient from the nonlinearity of (2.1) through the transformation $u=\psi\exp(V/2)$, and then to apply Theorem 3.

3 Existence of inertial manifolds for the Smoluchowski equation

3.1 Transformed equation

We would like to transform the Smoluchowski equation in a manner that would eliminate the gradient from the nonlinear term. It can be easily verified that functions ψ and V satisfy (2.1) if and only if $u=\psi\exp(V/2)$ and V satisfy the equation

$$\partial_t u=\Delta_{\mathbf{m}}u+\frac{1}{2}\left(\partial_t V+\Delta_{\mathbf{m}}V-\frac{1}{2}|\nabla_{\mathbf{m}}V|^2\right)u.$$

However, due to the dependence of V on ψ, this is not a closed equation in u, and it turns out that it is not possible to express V as a function of u. We can circumvent this difficulty by first performing the following transformation:

$$\xi = \Theta(\psi) = \psi - 2P_2\psi + d = \psi + cV + d,$$

where $c = \frac{15}{4b\pi}$ and $d = \frac{5}{2\pi}$. Notice that $\psi > 0$ implies $\xi > 0$ and $\int_{S^2}\psi = 1$ implies $\int_{S^2}\xi = 11$. It can be easily seen that ψ satisfies the Smoluchowski equation if and only if ξ satisfies

$$\partial_t\xi = \Delta_{\mathbf{m}}\xi + \nabla_{\mathbf{m}}\cdot(\xi\nabla_{\mathbf{m}}V) + c(\partial_t V - \Delta_{\mathbf{m}}V - |\nabla_{\mathbf{m}}V|^2 - V\Delta_{\mathbf{m}}V) - d\Delta_{\mathbf{m}}V, \tag{3.11}$$

where $V = \frac{b}{2}\mathbf{w}\cdot < \mathbf{w} >_\xi = -\frac{b}{2}\mathbf{w}\cdot < \mathbf{w} >_\psi$. Similarly as above, this is equivalent to $u = \xi\exp(V/2)$ satisfying the equation

$$\begin{aligned}\partial_t u = \Delta_{\mathbf{m}}u &+ e^{V/2}[\tfrac{1}{2}\left(\partial_t V + \Delta_{\mathbf{m}}V - \tfrac{1}{2}|\nabla_{\mathbf{m}}V|^2\right)\xi \\ &+ (c(\partial_t V - \Delta_{\mathbf{m}}V - |\nabla_{\mathbf{m}}V|^2 - V\Delta_{\mathbf{m}}V) - d\Delta_{\mathbf{m}}V)].\end{aligned} \tag{3.12}$$

The equation (2.5) for the evolution of V and the fact that $\Delta_{\mathbf{m}}V = -6V$ allow us to rewrite this equation in the form

$$\partial_t u = \Delta_{\mathbf{m}}u + F(\xi), \tag{3.13}$$

where $F(\xi): S^2 \to \mathbb{R}$ depends Lipschitz-continuously on $\mathbf{m}$, ξ and the second and fourth moments of ξ. Our next goal is to express V and therefore also ξ as a function of u in order to view (3.13) as a closed semilinear parabolic equation in u. To this end, we develop the following framework. If $u \in L^1(S^2)$, we define the transform $\widehat{u} \in C^\infty(\mathbb{R}^2)$

$$\widehat{u}(\mathbf{x}) = \int_{S^2} u(\mathbf{m})e^{-\mathbf{x}\cdot\mathbf{w}(\mathbf{m})}\, d\mathbf{m}.$$

For $\mathbf{a} \in \mathbb{R}^2$ we define

$$\mu_{\mathbf{a}}u(\mathbf{m}) := u(\mathbf{m})e^{-\mathbf{a}\cdot\mathbf{w}(\mathbf{m})} \in L^1(S^2),$$

and so

$$\widehat{\mu_{\mathbf{a}}u}(\mathbf{x}) = \int_{S^2} u(\varphi)e^{-(\mathbf{x}+\mathbf{a})\mathbf{w}(m)}\, d\mathbf{m} =: \tau_{\mathbf{a}}\widehat{u}(\mathbf{x}).$$

We define the function spaces

$$X = \left\{\xi \in L^2(S^2;\mathbb{R}^+) : \xi \text{ symmetric}, < \mathbf{w} >_\xi \neq \mathbf{o}, \int_{S^2}\xi = 11\right\}$$

and

$$\mathcal{H} = \Big\{ u \in L^2(S^2; \mathbb{R}^+) : u \text{ symmetric} \Big\}.$$

Also, let

$$\mathcal{X} = \Big\{ u \in \mathcal{H} : \int_{S^2} u > 11, \int_{S^2} \mu_{\mathbf{a}} u < 11 \text{ for some } \mathbf{a} \in \mathbb{R}^2 \Big\},$$

which is an open subset of $\mathcal{H}$. For $u \in \mathcal{X}$ we have

$$\nabla \hat{u}(\mathbf{x}) = -\int_{S^2} \mu_{\mathbf{x}} u(\mathbf{m}) \mathbf{w}(\mathbf{m}) \; d\mathbf{m},$$

$$\nabla\nabla \hat{u}(\mathbf{x}) = \int_{S^2} \mu_{\mathbf{x}} u(\mathbf{m}) (\mathbf{w}(\mathbf{m})\mathbf{w}(\mathbf{m})) \; d\mathbf{m}.$$

$\nabla\nabla\hat{u}(\mathbf{x})$ is positive definite, since by Cauchy-Schwarz

$$\det(\nabla\nabla\hat{u}) = < w_1^2 >< w_2^2 > - < w_1 w_2 >^2 > 0,$$

where $< f > = \int_{S^2} f(\mathbf{m}) \mu_x u(\mathbf{m}) \; d\mathbf{m}$, so $\hat{u}$ is a convex up. Since $u \in \mathcal{X}$, the level set

$$\Gamma(u) = \{\mathbf{x} \in \mathbb{R}^2 | \hat{u}(\mathbf{x}) \leq 11\}$$

is nonempty convex set, and the point $\mathbf{o} = (0,0) \notin \Gamma(u)$. Thus, there exists a unique point $\mathbf{r} \in \partial\Gamma(u)$ so that $|\mathbf{r}| = \operatorname{dist}(\Gamma(u), \mathbf{o})$. Note that $\mathbf{r}$ is the unique point on $\partial\Gamma(u)$ for which there exists $b > 0$ such that

$$\nabla\hat{u}(\mathbf{r}) = -\frac{4}{b}\mathbf{r}.$$

We now define the mappings

$$\begin{aligned}
&R : \mathcal{X} \to \mathbb{R}^2, \\
&\quad u \mapsto \mathbf{r}, \\
&\Xi : \mathcal{X} \to X, \\
&\quad u \mapsto \xi = \mu_{R(u)} u = u e^{-R(u)\cdot\mathbf{w}}, \\
&Y : \mathcal{X} \to \mathbb{R}^2, \\
&\quad u \mapsto -(\nabla\hat{u})(R(u)) = \int_{S^2} u(\mathbf{m}) e^{-R(u)\cdot\mathbf{w}(\mathbf{m})} \mathbf{w}(\mathbf{m}) \; d\mathbf{m} = < \mathbf{w} >_{\Xi(u)} \\
&B : \mathcal{X} \to \mathbb{R}^+, \\
&\quad u \mapsto b = 4|R(u)|/|Y(u)|.
\end{aligned}$$

Note the inequality $R(u) \leq B(u)/4$. We will need the following:

Lemma 3 *R, Ξ, Y, and B are continuous functions on $\mathcal{X}$.*

Proof : We prove the continuity of R, and the continuities of Ξ, Y and B follow. To prove the statement by contradiction, we chose a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{X}$ and $u\in\mathcal{X}$ such that $v_n\to u$ in $L^2(S^2)$. This obviously implies implies $\widehat{v_n}\to\widehat{u}$ and $\widehat{v_n}'\to\widehat{u}'$ in $L^\infty(S^2)$. Let $\mathbf{r}=R(u)$, $\mathbf{s}_n=R(v_n)$, and suppose $\mathbf{s}_n\not\to\mathbf{r}$ as $n\to\infty$. Let $b_n=4|R(v_n)|/|Y(v_n)|=4|\mathbf{s}_n|/|\nabla\widehat{v}_n(\mathbf{s}_n)|$. One can easily observe that the sequence $(\mathbf{s}_n)$ is bounded. Therefore, without loss of generality, we can assume that $\mathbf{s}_n\to\mathbf{s}\neq\mathbf{r}$ as $n\to\infty$. Because of the convergence in the sup norm, $\widehat{v_n}(\mathbf{s}_n)\to\widehat{u}(\mathbf{s})$ and $\nabla\widehat{v_n}(\mathbf{s}_n)\to\nabla\widehat{u}(\mathbf{s})$. Therefore, $\widehat{v_n}(\mathbf{s}_n)=11$ implies $\widehat{u}(\mathbf{s})=11$, and $b_n\to b:=4|\mathbf{s}|/|\nabla\widehat{u}(\mathbf{s})|$, so $\nabla\widehat{u}(\mathbf{s})=-\frac{4}{b}\mathbf{s}$. This is a contradiction to $\mathbf{s}\neq\mathbf{r}$. $\square$

Corollary 1 *Let $b>0$ be fixed, and let $\mathcal{X}_b=B^{-1}\{b\}$. The function*

$$\Xi_b=\Xi|_{\mathcal{X}_b}:\mathcal{X}_b\to X$$

is a homeomorphism, and its inverse is given by

$$\Xi_b^{-1}(\xi)=\xi e^{(b/4)<\mathbf{w}>_\xi\cdot\mathbf{w}}.$$

As already mentioned, $\xi(t)=\Theta(\psi(t))$ is a solution of (3.11) for some $b>0$, if and only if

$$u(t)=\xi(t)e^{V(t)/2}=\xi(t)e^{(b/4)<\mathbf{w}>_{\xi(t)}\cdot\mathbf{w}}=\Xi_b^{-1}(\xi(t))$$

satisfies (3.13). Denoting by

$$\mathbf{r}(t)=\frac{b}{4}<\mathbf{w}>_{\xi(t)},$$

we have $\xi(t)=u(t)\exp(-\mathbf{r}(t)\cdot\mathbf{w})$. Then, $\int_{S^2}\xi(\mathbf{m},t)\,d\mathbf{m}=11$ implies $\widehat{u}(\mathbf{r}(t))=11$, and multiplying by $\mathbf{w}$ and integrating over S^2, we obtain $\nabla\widehat{u}(\mathbf{r}(t))=-\frac{4}{b}\mathbf{r}(t)$. Using framework developed earlier, we conclude that $\mathbf{r}(t)=R(u(t))$, $\xi(t)=\Xi(u(t))$ and $b=B(u(t))$, $t\geq 0$. Therefore, $u(t)$ satisfies the closed equation

$$u_t=\Delta_{\mathbf{m}}u+F(\Xi(u)). \tag{3.14}$$

On the other hand, if $u(t)\in\mathcal{X}$, $t\geq 0$ is a solution to (3.14) for an initial datum $u_0=\Xi_b^{-1}(\Theta(\psi_0))$ for some $\psi_0\in H$, it is immediate that $\psi(t)=\Theta^{-1}\Xi(u(t))$ satisfies (2.1), and $b=B(u(t))$ is preserved by the solution operator.

3.2 Prepared equation

In order to apply the classical theory, we need the nonlinear term in (3.14) to satisfy a Lipschitz condition. Since this is not necessarily true, we have to modify the equation (3.14) in a way that preserves its long-term behavior. This is done by modifying the nonlinear term outside of an absorbing set. The equation obtained in this way is referred to as the prepared equation. In order to see how we are to define the prepared equation, let us examine the Lipschitz continuity of the above defined functions.

Lemma 4 *The functions* $R|_{\mathcal{X}_b}$, $\Xi|_{\mathcal{X}_b}$, $Y|_{\mathcal{X}_b}$, *and* $F\circ\Xi|_{\mathcal{X}_b}$ *are Lipschitz continuous. In particular,* $\Xi_b:\mathcal{X}_b\to X$ *is a Lipschitz homeomorphism.*

Proof : We prove the Lipschitz continuity of R, and the others follow. Let $u,v\in\mathcal{X}_b$, and let $\mathbf{r}=R(u)$, $\mathbf{s}=R(v)$. The mean-value theorem implies the existence of $\theta_1\in[0,1]$ and $\theta_2\in[0,1]$ so that, with the convexity of $\widehat{u}$ and $\widehat{v}$, we have

$$\widehat{u}(\mathbf{s})-\widehat{u}(\mathbf{r})=\nabla\widehat{u}(\mathbf{r}+\theta_1(\mathbf{s}-\mathbf{r}))\cdot(\mathbf{s}-\mathbf{r})\geq\nabla\widehat{u}(\mathbf{r})\cdot(\mathbf{s}-\mathbf{r})=-\frac{4}{b}\mathbf{r}(\mathbf{s}-\mathbf{r})$$

and

$$\widehat{v}(\mathbf{r})-\widehat{v}(\mathbf{s})=\nabla\widehat{v}(\mathbf{s}+\theta_2(\mathbf{r}-\mathbf{s}))\cdot(\mathbf{r}-\mathbf{s})\geq\nabla\widehat{v}(\mathbf{s})\cdot(\mathbf{r}-\mathbf{s})=-\frac{4}{b}\mathbf{s}(\mathbf{r}-\mathbf{s}).$$

Adding both equations yields

$$\int_{S^2}(u(\mathbf{m})-v(\mathbf{m}))(e^{-\mathbf{s}\cdot\mathbf{w}(\mathbf{m})}-e^{-\mathbf{r}\cdot\mathbf{w}(\mathbf{m})})\ d\mathbf{m}\geq\frac{4}{b}|\mathbf{s}-\mathbf{r}|^2,$$

and therefore there exists $C_3=C_3(b)$ so that

$$\frac{4}{b}|\mathbf{s}-\mathbf{r}|^2\leq C_3\,\|u-v\|_2\,|\mathbf{r}-\mathbf{s}|,$$

and so $|R(u)-R(v)|\leq\frac{bC_3}{4}\,\|u-v\|_2$. □

For technical reasons we will need the following

Lemma 5 *Let*

$$\mathcal{B}_{\kappa(b)}=\{u\in\mathcal{H}:\|u\|_2<\kappa(b)\}\supset\Xi_b^{-1}(\Theta(B_{\rho(b)}))$$

We chose $\kappa:\mathbb{R}^+\to\mathbb{R}^+$ *to be continuos and increasing. Let* $b_1>0$ *and* $r_1>0$. *Let* $\mathcal{B}=\mathcal{B}_{\kappa(b_1)}\cap B^{-1}(0,b_1)\cap R^{-1}\{\mathbf{r}\in\mathbb{R}^2:r_1<|\mathbf{r}|\}$. *Then* $R|_{\mathcal{B}}$, $\Xi|_{\mathcal{B}}$, $Y|_{\mathcal{B}}$, $B|_{\mathcal{B}}$ *and* $F\circ\Xi|_{\mathcal{B}}$ *are Lipschitz continuous.*

Proof : Let $u,v \in \mathcal{B}$, and let $\mathbf{r}=R(u)$, $\mathbf{s}=R(v)$. As before, we have

$$\widehat{u}(\mathbf{s})-\widehat{u}(\mathbf{r}) \geq \frac{4}{B(u)}\mathbf{r}(\mathbf{r}-\mathbf{s})$$

and

$$\widehat{v}(\mathbf{r})-\widehat{v}(\mathbf{s}) \geq \frac{4}{B(v)}\mathbf{s}(\mathbf{s}-\mathbf{r}).$$

Since $\mathbf{r}(\mathbf{r}-\mathbf{s})+\mathbf{s}(\mathbf{s}-\mathbf{r})=|\mathbf{r}-\mathbf{s}|^2 \geq 0$, we distinguish the following cases:
Case 1: $\mathbf{r}(\mathbf{r}-\mathbf{s}) \geq 0$ and $\mathbf{s}(\mathbf{s}-\mathbf{r}) \geq 0$.
In this case, similarly as in the previous Lemma, we have

$$\int_{S^2}(u(\mathbf{m})-v(\mathbf{m}))(e^{-\mathbf{s}\cdot\mathbf{w}(\mathbf{m})}-e^{-\mathbf{r}\cdot\mathbf{w}(\mathbf{m})})\ d\mathbf{m} \geq \frac{4}{b_2}|\mathbf{s}-\mathbf{r}|^2,$$

and so $|R(u)-R(v)| \leq \frac{b_1 C_3(b_1)}{4}\|u-v\|_2$.
Case 2: $\mathbf{r}(\mathbf{r}-\mathbf{s})<0$ and $\mathbf{s} \notin \Gamma(u)$.
In this case,

$$\widehat{u}(\mathbf{s})-\widehat{u}(\mathbf{r})>0>\frac{4}{B(v)}\mathbf{r}(\mathbf{r}-\mathbf{s})$$

and

$$\widehat{v}(\mathbf{r})-\widehat{v}(\mathbf{s}) \geq \frac{4}{B(v)}\mathbf{s}(\mathbf{s}-\mathbf{r}),$$

and one arrives at the same conclusion as in the previous case.
Case 3: $\mathbf{r}(\mathbf{r}-\mathbf{s})<0$ and $\mathbf{s} \in \Gamma(u)$.
Since $\mathbf{s} \in \Gamma(u)$, there exists $\mathbf{s}' \in \partial\Gamma(u) \cap [\mathbf{o},\mathbf{s}]$. Let $v'=\mu_{\mathbf{s}-\mathbf{s}'}v$, and so $\widehat{v}'=\tau_{\mathbf{s}-\mathbf{s}'}\widehat{v}$. Thus, $\widehat{v}'(\mathbf{s}')=\widehat{v}(\mathbf{s})=11$, so $R(v')=\mathbf{s}'$ follows. Another easy observation is that $\mathbf{r}(\mathbf{r}-\mathbf{s}') \leq 0$, and so

$$\widehat{u}(\mathbf{s}')-\widehat{u}(\mathbf{r})=0 \geq \frac{4}{B(v)}\mathbf{r}(\mathbf{r}-\mathbf{s}')$$

and

$$\widehat{v}'(\mathbf{r})-\widehat{v}'(\mathbf{s}') \geq \frac{4}{B(v')}\mathbf{s}'(\mathbf{s}'-\mathbf{r}) \geq \frac{4}{B(v)}\mathbf{s}'(\mathbf{s}'-\mathbf{r}),$$

and we conclude again $|\mathbf{r}-\mathbf{s}'| \leq \frac{b_1 C_3(b_1)}{4}\|u-v'\|_2$. On the other hand, since $\mathbf{s}$ and $\mathbf{s}'$ are collinear,

$$\widehat{v}(\mathbf{s}')-\widehat{v}(\mathbf{s}) \geq \frac{4}{B(v)}\mathbf{s}(\mathbf{s}-\mathbf{s}')=\frac{4}{B(v)}|\mathbf{s}||\mathbf{s}-\mathbf{s}'| \geq \frac{4r_1}{b_2}|\mathbf{s}-\mathbf{s}'|.$$

Since $\widehat{v}(\mathbf{s})=\widehat{u}(\mathbf{s}')$, we have $|\mathbf{s}-\mathbf{s}'|\le\frac{b_1C_3(b_1)}{4r_1}\|v-u\|_2$. The desired follows with the estimate

$$\begin{aligned}\|v-v'\|_2^2 &= \int_{S^2}(v(\mathbf{m})-v'(\mathbf{m}))^2 d\mathbf{m} = \int_{S^2} v(\mathbf{m})^2(1-e^{(\mathbf{s}'-\mathbf{s})\cdot\mathbf{w}(\mathbf{m})})^2\, d\mathbf{m}\\ &= \int_{S^2}[\Xi(v)(\mathbf{m})]^2(e^{\mathbf{s}\cdot\mathbf{w}(\mathbf{m})}-e^{\mathbf{s}'\cdot\mathbf{w}(\mathbf{m})})^2\, d\mathbf{m} \le C_4^2|\mathbf{s}-\mathbf{s}'|^2\,\|\Xi(v)\|_2^2 \le C_5^2\,\|v-u\|^2.\end{aligned}$$

where the constants C_4 and C_5 depend on r_1 and b_1 only.
Case 4: $\mathbf{s}(\mathbf{s}-\mathbf{r})<0$.
The inequalities for this case follow in an analogue fashion to the previous two cases. □

Lemma 6 *Let $0<b<b_1$ and $r_1>r^*(b)$, and let $\mathcal{B}$ be defined as in the previous Lemma. Let $\mathcal{B}'=\mathcal{B}\cup(\mathcal{X}_b\cap\mathcal{B}_{\kappa(b)})$. Then the functions $R|_{\mathcal{B}'}$, $\Xi|_{\mathcal{B}'}$, $Y|_{\mathcal{B}'}$, $B|_{\mathcal{B}'}$ and $F\circ\Xi|_{\mathcal{B}'}$ are Lipschitz continuous.*

Proof : Let us partition $\mathcal{B}'$ into three regions: $\mathcal{B}_1=\mathcal{X}_b\cap\mathcal{B}_{\kappa(b)}\cap R^{-1}(0,r_1/2]$, $\mathcal{B}_2=\mathcal{X}_b\cap\mathcal{B}_{\kappa(b)}\cap R^{-1}(r_1/2,r_1)$, and $\mathcal{B}$. By the previous two Lemmas, the functions are Lipschitz continuous on any of these three regions, as well as on sets $\mathcal{B}_1\cup\mathcal{B}_2$ and $\mathcal{B}_2\cup\mathcal{B}$. The Lipschitz continuity on $\mathcal{B}_1\cup\mathcal{B}$ follows since both of these sets are bounded in L_2 and $\operatorname{dist}(R(\mathcal{B}_1),R(\mathcal{B}))>r_1/2$. This implies the Lipschitz continuity on $\mathcal{B}'$. □

We can now define the prepared equation:

$$N_P(u)=\begin{cases}F[\Xi(u)], & \text{if } u\in\mathcal{B}'\\ 0, & \text{if } u\in\mathcal{H}\backslash\mathcal{B}_{2\kappa(b_1)}\end{cases}$$

This is clearly a Lipschitz function on $\mathcal{B}'\cup(\mathcal{H}\backslash\mathcal{B}_{2\kappa(b_1)})$. Denote by $C>0$ its Lipschitz constant. It is a well known fact from analysis that a Lipschitz continuous function defined on a subset of a Hilbert space can be extended to a Lipschitz continuous function defined on the entire Hilbert space, even preserving the Lipschitz constant $C>0$. Without changing the notation, let $N_P:\mathcal{H}\to\mathrm{I\!R}$ be such an extension. The prepared equation reads now

$$\partial_t u + Au = N_P(u).$$

3.3 Main theorem

We are now in a position to prove the existence of the inertial manifold of the Smoluchowski equation (2.1).

Theorem 4 *Let $0<b\neq \frac{15}{2}$. The Smoluchowski equation on the unit sphere with the Maier-Saupe potential possesses an asymptotically complete inertial manifold $\mathcal{M}_b$.*

Proof : The positivity of A and the Lipschitz continuity of N_P ensure that the prepared equation generates a strongly continuous semigroup $S_P(t)$. The fact that N_P vanishes outside of $B_{2\kappa(b_1)}$ suffices to prove that the prepared equation is dissipative, and that it possesses a finite-dimensional global attractor $\mathcal{A}_P$. Also, by construction, $\Xi_b^{-1}(\Theta(\mathcal{A}))\subset\mathcal{A}_P$. There exists $n\in\mathbb{N}$ such that $\lambda_{n+1}-\lambda_n=2n+2>4C$, and the spectral gap condition is satisfied for the prepared equation. The Theorem 3 applies, and we infer the existence of an asymptotically complete inertial manifold $\mathcal{M}_P\supset\mathcal{A}_P$ for the prepared equation given as a graph of a Lipschitz function Φ_P:

$$\mathcal{M}_P=\mathcal{G}[\Phi_P]=\{p+\Phi_P(p):p\in P_n\mathcal{H}\}.$$

We now define $\mathcal{M}_b=B_{\rho(b)}\cap\Theta^{-1}(\Xi(\mathcal{M}_P))$. Since $\Xi_b:\mathcal{X}_b\to X$ is a Lipschitz homeomorphism, it is immediate that $\mathcal{M}_b$ is a finite dimensional Lipschitz manifold. It is positively invariant under $S(t)$, since both $B_{\rho(b)}$ and $\Theta^{-1}(\Xi_b(\mathcal{M}_P))$ are positively invariant. It remains to prove that $\mathcal{M}_b$ is exponentially attracting and asymptotically complete. Let $\psi_0\in H$ and $\psi(t)=S(t)\psi_0$. Let $u(t)=\Xi_b^{-1}(\Theta(\psi(t)))$, $t\geq 0$. Since the convergence to the isotropic steady-state $\bar{\psi}=1/4\pi$ is always exponential if $b\neq 15/2$, we can assume without loss of generality that $\psi(t)\to\psi^*$ as $t\to\infty$ for an anisotropic state ψ^*. Let $u^*=\Xi_b^{-1}(\Theta(\psi^*))$, so $u(t)\to u^*$ as $t\to\infty$. On the other hand, since $\mathcal{M}_P$ is exponentially attracting and asymptotically complete, there exists $v_0\in\mathcal{M}_P$ so that for $v_P(t)=S_P(t)v_0$ we have $\|u(t)-v_P(t)\|_2\to 0$, as $t\to\infty$, exponentially. Thus, $v_P(t)\to u^*$ as $t\to\infty$ as well. Since $u^*\in\mathcal{B}$, there exists $T>0$ so that $v_P(t)\in\mathcal{B}$ for $t\geq T$. However, since $N_P|_{\mathcal{B}}=N|_{\mathcal{B}}$, $B(v_P(t))=B(u^*)=b$ for $t\geq T$. Therefore, $\sigma(t):=\Theta^{-1}(\Xi(v_P(t)))\in\Xi(\mathcal{M}_P)$, $t\geq T$ is a solution of (2.1). For some $T'\geq T$ we have $\sigma(t)\in B_{\rho(b)}$, $t\geq T'$, and therefore $\sigma(t)\in\mathcal{M}_b$, $t\geq T'$. Since Ξ_b is Lipschitz continuous, $\|\psi(t)-\sigma(t)\|_2\to 0$, as $t\to\infty$, exponentially. This concludes the proof. $\square$

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