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On the Backwards Behavior of the Solutions of the 2D Periodic Viscous Camassa–Holm Equations

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The global behavior of the periodic 2D viscous Camassa–Holm equations is studied. The set of initial data for which the solution exists for all negative times and grows backwards with an exponential rate no greater then some given value is observed and proved to possess some interesting richness and density properties.

KEY WORDS: Viscous Camassa–Holm equations; vorticity quotients; global solutions.

1. INTRODUCTION

In this paper we consider the two dimensional viscous Camassa–Holm equations

$$
\begin{gathered}
\frac{d}{dt} v - \nu \, \Delta v + (u \cdot \nabla) \, v + v_j \, \nabla u_j + \nabla p = f \\
v = u - \alpha^2 \, \Delta u \\
\nabla \cdot u = 0
\end{gathered}
\tag{1.1}
$$

where f is a time independent force, and $\nu > 0$ (representing the kinematic viscosity) and $\alpha > 0$ are given parameters. The functions u: $\mathbb{R}^2 \to \mathbb{R}^2$, v: $\mathbb{R}^2 \to \mathbb{R}^2$ and p: $\mathbb{R}^2 \to \mathbb{R}$ are unknowns representing velocity, momentum, and modified pressure, respectively. We will refer to the equations (1.1) as (VCHE).

(VCHE) is a generalization of an one-dimensional equation derived by Camassa and Holm (1993), which describes unidirectional surface waves in

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shallow water. Holm *et al.* (1998) generalized the equations to n dimensions, the so called ideal Camassa–Holm equations or Euler alpha model. The parameter α is interpreted as the typical mean amplitude of the fluctuations.

Since v in (1.1) represents a momentum, it is plausible to let viscosity act to diffuse this momentum. Chen *et al.* (1999) proposed the viscous variant of the Camassa–Holm equations in which an artificial viscosity term $\nu\,\Delta v$ is introduced into the system. The (VCHE) closely resemble the Navier–Stokes equations (NSE):

$$\frac{d}{dt}u - \nu\,\Delta u + (u\cdot\nabla)\,u + \nabla p = f$$

$$\nabla\cdot u = 0$$

and even coincide with them for $\alpha = 0$. For this reason they are also known as Navier–Stokes-α model.

In this paper we are interested in the backwards behavior of the solutions of the (VCHE). We will study the set $\mathscr{G}$ of initial data u_0 for the (VCHE) for which the solution $S(t)\,u_0$ exists for all times $t \in \mathbb{R}$ and the rate of the backwards growth for those solutions. The backwards behavior was studied for other related partial and ordinary differential equations with a similar structure as the (VCHE) and the (NSE). Some examples are the Kuramoto–Sivashinsky equation (see Kukavica (1992)), and the Ginzburg–Landau equation (see Doering *et al.* (1988)). For a global solution of those equations it was shown that if it grows at most exponentially when $t \to -\infty$, then it is necessarily uniformly bounded. Constantin *et al.* (1997) studied the backwards behavior for the periodic 2D Navier–Stokes equations and proved that it demonstrates more similarity with the linear case. For example, it was proved that the set of initial data for which the solution exists for all negative times and grows at most exponentially when $t \to -\infty$ is dense in the phase space of the (NSE). In this paper we will show that the backwards behavior for the periodic 2D (VCHE) resembles the (NSE) case.

The major object of study in this paper is the set

$$\mathscr{M}_n = \mathscr{A} \cup \left\{ u_0 \in \mathscr{G}\setminus\mathscr{A} : \lim_{t\to-\infty} \frac{|A^{1/2}(I+\alpha^2 A)\,S(t)\,u_0|^2}{|(I+\alpha^2 A)^{1/2}\,S(t)\,u_0|^2} \leqslant \lambda_n(1+\alpha^2\lambda_n) \right\}$$

where $|\cdot|$ denotes the usual L^2 norm, A denotes the Stokes operator, and $0 < \lambda_1 < \lambda_2 < \cdots$ its distinct eigenvalues. $\mathscr{A}$ stands for the global attractor for the (VCHE). For convenience, we will refer to the quotient in the definition of $\mathscr{M}_n$ as the vorticity quotient. It emerges naturally from the "energy" and "enstrophy" balance equations for the (VCHE) and is an analogue of the Dirichlet quotient for the (NSE), which was studied in the paper by

Constantin *et al.* (1997). Most of the results on Dirichlet quotients have an analogue in the present paper, but there are differences in the proofs due to the different structure of the nonlinear terms in the (VCHE). Equivalently, we can define the set $\mathscr{M}_n$ in the following way:

$$\mathscr{M}_n = \{u_0 \in \mathscr{G}\colon |(I+\alpha^2 A)^{1/2} S(t)\, u_0| = \mathscr{O}(e^{\nu\lambda_n |t|}) \text{ as } t \to -\infty\}$$

In other words, $\mathscr{M}_n$ consists of precisely those initial data, for which the solution $S(t)\, u_0$ exists for all $t \in \mathbb{R}$, and its kinetic energy $|(I+\alpha^2 A)^{1/2} S(t)\, u_0|^2$ increases with an exponential growth rate no greater then $\nu\lambda_n$ when $t \to -\infty$.

In Section 2 we recall some known facts about the (VCHE), and we rewrite it in its functional form. We derive the "energy" and "enstrophy" balance equations and some basic inequalities that emerge from them. In this section, we also define the set $\mathscr{G}$ consisting of initial data for which there exists a global solution, as well as the sets $\mathscr{M}_n$. In Section 3 we prove some useful properties of the sets $\mathscr{M}_n$. The central result of this section is Theorem 1, which enables us to control the vorticity quotients of a solution of (VCHE) by controlling the energy of the solution from below. This result is a useful technical tool and will be used to obtain all the other results. Another basic result of this section is Theorem 3 that provides us with a tool for producing global solutions with certain properties. In Section 4 we prove that the sets $\mathscr{M}_n$ are rich in the sense that $P_n \mathscr{M}_n = P_n H$ (H being a suitable Hilbert space defined in Section 1, and $P_n H$ being the spectral space of A corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$). In Section 5 we use this result to prove some density properties of the sets $\mathscr{M}_n$. In Theorem 6 we prove that the set of initial data for which there exists a global solution, which grows at most exponentially, is dense in H. We conclude the paper with a result in which we show that the eigenvectors of the Stokes operator A corresponding to λ_n for some $n \in \mathbb{N}$ belong to the closure of scaled set $\mathscr{M}_n$, even in respect with the stronger "energy" norm of the (VCHE). The analogue of this result for the (NSE) has been proved by C. Foias and I. Kukavica, and is yet to be published.

2. PRELIMINARIES

We consider the two dimensional viscous Camassa–Holm equations

$$\frac{d}{dt} v - \nu\, \Delta v + (u \cdot \nabla)\, v + \sum_{j=1}^{2} v_j\, \nabla u_j + \nabla p = f$$

$$v = u - \alpha^2\, \Delta u$$

$$\nabla \cdot u = 0$$

where $f \in L^2(\Omega)^2$ (which is Ω-periodic and $\int_\Omega f = 0$) is a given function representing body forcing, $\nu > 0$ is the constant viscosity, and $\alpha > 0$ is a given parameter. u: $\mathbb{R}^2 \to \mathbb{R}^2$, v: $\mathbb{R}^2 \to \mathbb{R}^2$ and p: $\mathbb{R}^2 \to \mathbb{R}$, are unknown functions representing velocity, vorticity and modified pressure, respectively. We supplement the system with the periodic boundary condition

$$u(x+Le_j) = u(x), \qquad x \in \mathbb{R}^2, \quad 1 \leqslant j \leqslant 2$$

and

$$\int_\Omega u = 0$$

where $L > 0$, $\Omega = [0, L]^2$, and $(e_j)_{j=1}^2$ is the canonical basis in $\mathbb{R}^2$. Since we are interested in a functional setting for these equations, we need to introduce suitable functional spaces. First, we define

$$\mathscr{V} = \Big\{ u : u \text{ is a vector valued } \Omega\text{-periodic trigonometric}$$

$$\text{polynomial, } \nabla \cdot u = 0, \int_\Omega u = 0 \Big\}$$

We define H and V to be closures of $\mathscr{V}$ in the (real) Hilbert spaces $L^2(\Omega)^2$ and $H^1(\Omega)^2$, respectively. H and V are also (real) Hilbert spaces with respective scalar products

$$(u, v) = \sum_{j=1}^{2} \int_\Omega u_j v_j, \qquad u, v \in H$$

and

$$((u, v)) = \sum_{j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_k} \frac{\partial v_j}{\partial x_k}, \qquad u, v \in V$$

The corresponding norms are $|u| = (u, u)^{1/2}$ for $u \in H$ and $\|u\| = ((u, u))^{1/2}$ for $u \in V$ respectively. We denote the dual of V by V'. By the Rellich embedding theorem, the natural inclusions $i_1 : V \to H$ and $i_2 : H \to V'$ are compact.

We denote the orthogonal projection (called the Leray-Projector) on the space H by $P_L : L^2(\Omega)^2 \to H$; observe that $H^\perp = \{\nabla p : p \in H^1(\Omega)\}$ is the orthogonal complement of H in $L^2(\Omega)$. By $A = -P_L \Delta$ we denote the Stokes operator with domain $D(A) = H^2(\Omega)^2 \cap V$. A is a selfadjoint positive operator with compact inverse. The eigenvalues of A are of the form $(k_1^2+k_2^2)(2\pi/L)^2$ where $k_1, k_2 \in \mathbb{N}_0$, and $k_1^2+k_2^2 \neq 0$. We can arrange them in increasing order

$$\left(\frac{2\pi}{L}\right)^2 = \lambda_1 < \lambda_2 < \cdots$$

Note that

$$\lim_{n\to\infty} \lambda_n = \infty \tag{2.1}$$

and

$$\lambda_{n+1} - \lambda_n \geqslant \lambda_1$$

For the purposes of this paper, an important property is also

$$\limsup_{n\to\infty} (\lambda_{n+1} - \lambda_n) = \infty \tag{2.2}$$

We define P_n to be the orthogonal projection in H on the spectral space of A corresponding to the eigenvalues $\lambda_1, \lambda_2,\ldots, \lambda_n$; also let $Q_n := I - P_n$. Integration by parts gives

$$\|u\| = |A^{1/2}u|, \qquad u \in V$$

We also define

$$[u] := |(I+\alpha^2 A)^{1/2}\, u|, \qquad u \in V$$

and

$$|[u]| := |A^{1/2}(I+\alpha^2 A)^{1/2}\, u|, \qquad u \in D(A)$$

We will use the Poincaré inequality

$$\|u\|^2 \geqslant \lambda_1\, |u|^2, \qquad u \in V$$

and, similarly,

$$[u]^2 \geqslant (1+\alpha^2\lambda_1)\, |u|^2, \qquad u \in V$$

Regarding the nonlinear terms that appear in the Camassa–Holm equations, we introduce the following bilinear forms:

$$B(u, v) := P_L((u\cdot\nabla)\, v), \qquad u \in H, \quad v \in V$$

and

$$B^*(u, v) := P_L\left(\sum_{j=1}^{2} v_j \nabla u_j\right), \qquad u \in V, \quad v \in H$$

By integrating by parts we obtain following important identities:

$$(B(u, v), w) = -(B(u, w), v), \qquad u \in H, \quad v, w \in V \tag{2.3}$$

and

$$(B(u, v), Av) = (B(Av, v), u), \qquad u \in H, \quad v \in D(A) \tag{2.4}$$

The connection between B and B^* is given by the identity

$$(B^*(v, w), u) = (B(u, v), w), \qquad u \in H, \quad v, w \in V \tag{2.5}$$

Now we can rewrite the viscous Camassa–Holm equations in the functional form

$$\begin{aligned} \dot{v} + \nu A v + B(u, v) + B^*(u, v) &= f \\ v &= (I + \alpha^2 A)\, u \end{aligned} \tag{2.6}$$

where $P_L f$ is replaced by $f \in H$. Observe that the term containing modified pressure doesn't occur in this equation since $P_L(\nabla p) = 0$. This new functional version of the (VCHE) is understood in V'. Classical theorems imply that, for every $u_0 \in V$, there exists a unique solution $u(t) = S(t)\, u_0$ for $t \geqslant 0$ of (2.6), which satisfies $u(0) = u_0$. If the solution $u(t)$ also exists for $t \in [-t_0, 0]$ for some $t_0 > 0$, then it is still uniquely determined by u_0, and we still denote it by $S(t)\, u_0$. It is known that $u \in L^\infty_{\mathrm{loc}}((0, \infty) : H^3(\Omega))$. Also, for any $t_0 > 0$, the solution operator $S(t_0)\colon V \to V$ is continuous.

Our aim is now to find balance equations for (VCHE), that do not involve nonlinear terms. To this end, observe that for $u, v \in V$

$$\begin{aligned} (B(u, v), u) + (B^*(u, v), u) &= (B(u, v), u) + (B(u, u), v) \\ &= (B(u, v), u) - (B(u, v), u) = 0 \end{aligned}$$

and for $u \in V$, $v \in D(A)$

$$\begin{aligned} (B(u, v), Av) + (B^*(u, v), Av) &= (B(u, v), Av) + (B(Av, u), v) \\ &= (B(u, v), Av) - (B(Av, v), u) = 0 \end{aligned}$$

Therefore, if we multiply (VCHE) by u and Av respectively, we obtain the following equations:

$$\frac{1}{2}\frac{d}{dt}[u]^2+\nu(Av, u)=(f, u) \tag{2.7}$$

and

$$\frac{1}{2}\frac{d}{dt}\|v\|^2+\nu(Av, Av)=(f, Av) \tag{2.8}$$

These equations yield, respectively, the following inequalities

$$\frac{d}{dt}[u]^2+\nu\lambda_1[u]^2\leqslant\frac{|f|^2}{\nu\lambda_1} \tag{2.9}$$

and

$$\frac{d}{dt}\|v\|^2+\nu\lambda_1\ \|v\|^2\leqslant\frac{|f|^2}{\nu} \tag{2.10}$$

Observe that from the relation (2.9), it follows that $t\to[u(\cdot)]$ is a decreasing function as long as $[u(t)]>|f|/\nu\lambda_1$. If u is a solution of (VCHE) defined on some interval $[t_0, \infty)$, the Gronwall lemma gives

$$[u(t)]^2\leqslant[u(t_0)]^2\, e^{-\nu\lambda_1(t-t_0)}+\frac{|f|^2}{\nu^2\lambda_1^2}(1-e^{-\nu\lambda_1(t-t_0)}),\qquad t\geqslant t_0 \tag{2.11}$$

and

$$\|v(t)\|^2\leqslant\|v(t_0)\|^2\, e^{-\nu\lambda_1(t-t_0)}+\frac{|f|^2}{\nu^2\lambda_1}(1-e^{-\nu\lambda_1(t-t_0)}),\qquad t\geqslant t_0 \tag{2.12}$$

Also, if the solution is defined on $[t, t_0]$ we have

$$[u(t)]^2\geqslant[u(t_0)]^2\, e^{\nu\lambda_1(t_0-t)}-\frac{|f|^2}{\nu^2\lambda_1^2}(e^{\nu\lambda_1(t_0-t)}-1) \tag{2.13}$$

Observe that, if the solution is defined on $(-\infty, t_0]$, the integral $\int_{-\infty}^{t_0}\frac{1}{[u(t)]^2}\,dt$ exists.

Every solution $u(t)=S(t)\,u_0$ of the (VCHE) for an initial datum $u_0\in V$, which is defined for all $t\in\mathbb{R}$, is called a global solution. u_0 belongs to a trajectory of a global solution if and only if $u_0 \in \bigcap_{t\geqslant 0} S(t)\,V$. Therefore, we define

$$\mathscr{G} := \bigcap_{t\geqslant 0} S(t)\,V \tag{2.14}$$

to be the set of initial data, for which there exists a global solution. The (VCHE) have a global attractor

$$\begin{aligned}\mathscr{A} &= \{u_0 \in \mathscr{G} : \limsup_{t\to-\infty}\,[S(t)\,u_0] < \infty\}\\ &= \left\{u_0 \in \mathscr{G} : [S(t)\,u_0] \leqslant \frac{|f|}{\nu\lambda_1},\ t\in\mathbb{R}\right\}\end{aligned}$$

$\mathscr{A}$ is $S(\cdot)$-invariant, meaning that $S(t)\,\mathscr{A} = \mathscr{A}$ for $t\geqslant 0$. Also, it is a nonempty, compact, connected subset of V. Moreover, every $S(\cdot)$-invariant set containing $\mathscr{A}$ is connected.

For $0<\kappa<\lambda_1/2$ let us define the sets

$$\begin{aligned}\mathscr{M}_{n,\kappa} := \mathscr{A} \cup \Bigg\{u_0 \in \mathscr{G}\setminus\mathscr{A} : \limsup_{t\to-\infty} \frac{\|(I+\alpha^2 A)\,S(t)\,u_0\|^2}{[S(t)\,u_0]^2}\\ \leqslant (\lambda_n+\kappa)(1+\alpha^2(\lambda_n+\kappa))\Bigg\}\end{aligned}$$

Some obvious properties of the sets $\mathscr{M}_{n,\kappa}$ are

$$S(t)\,\mathscr{M}_{n,\kappa} = \mathscr{M}_{n,\kappa}, \qquad t\geqslant 0, \quad n\in\mathbb{N}$$

and

$$\mathscr{A} \subset \mathscr{M}_{1,\kappa} \subset \mathscr{M}_{2,\kappa} \subset \cdots$$

In the next section, after proving a crucial theorem, we'll be able to prove some nontrivial properties of these sets. For example, we will prove that the set $\mathscr{M}_{n,\kappa}$ doesn't really depend on κ. This enables us to define a new set $\mathscr{M}_n$ as $\mathscr{M}_{n,\kappa}$ for an arbitrary choice of $0<\kappa<\lambda_1/2$.

For later purposes let us also define the following sets for $n\in\mathbb{N}$:

$$\mathscr{C}_{n,\kappa} := \left\{u_0 \in A^{-1}V : \frac{\|(I+\alpha^2 A)\,u_0\|^2}{[u_0]^2} \leqslant (\lambda_n+\kappa)(1+\alpha^2(\lambda_n+\kappa))\right\}$$

3. SOME PROPERTIES OF THE SETS $\mathscr{M}_{n,\kappa}$

In this section, we will prove some properties of the sets $\mathscr{M}_{n,\kappa}$. To this end, we will need the following crucial result:

Theorem 1. *Let $T > 0$ and $n \in \mathbb{N}$. Let u be a solution of (VCHE) for the initial datum $u_0 \in \mathscr{C}_{n,\kappa}$ and let $v_0 := (I + \alpha^2 A)\, u_0$. If*

$$[u(t)] > \frac{|f|}{\nu\kappa}, \qquad t \in [0, T) \tag{3.1}$$

then

$$\frac{\|v(t)\|^2}{[u(t)]^2} \leqslant (\lambda_n + \kappa)(1 + \alpha^2(\lambda_n + \kappa)), \qquad t \in [0, T]$$

or, in other words, $u(t) \in \mathscr{C}_{n,\kappa}$ for $t \in [0, T]$.

Proof. Let us denote

$$\tilde{u} := \frac{u}{[u]}, \qquad \tilde{v} := \frac{v}{[u]}$$

Applying (2.7) and (2.8) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\frac{\|v\|^2}{[u]^2} &= \frac{1}{[u]^4}\left(\frac{1}{2}\frac{d}{dt}\|v\|^2 \cdot [u]^2 - \|v\|^2 \cdot \frac{1}{2}\frac{d}{dt}[u]^2\right)\\ &= \frac{1}{[u]^2}\left[(-\nu(Av, Av) + (f, Av)) - \|\tilde{v}\|^2\,(-\nu(Av, u) + (f, u))\right]\\ &= -\nu(A\tilde{v}, A\tilde{v} - \|\tilde{v}\|^2\,\tilde{u}) + (f/[u], A\tilde{v} - \|\tilde{v}\|^2\,\tilde{u})\end{aligned}$$

Thus, we obtain

$$\frac{1}{2}\frac{d}{dt}\frac{\|v\|^2}{[u]^2} + \nu(A\tilde{v}, A\tilde{v} - \|\tilde{v}\|^2\,\tilde{u}) = (f/[u], A\tilde{v} - \|\tilde{v}\|^2\,\tilde{u}) \tag{3.2}$$

Let us define μ to be the positive solution of the quadratic equation

$$\|\tilde{v}\|^2 = \mu(1 + \alpha^2\mu) \tag{3.3}$$

Now observe that

$$\begin{aligned}(A\tilde{v}, A\tilde{v}-\|\tilde{v}\|^2\,\tilde{u}) &= (A\tilde{v}, A\tilde{v}-\mu(1+\alpha^2\mu)\,\tilde{u})\\ &= (A\tilde{v}, A\tilde{v}-\mu\tilde{v})+\mu(A\tilde{v}, \tilde{v}-(1+\alpha^2\mu)\,\tilde{u})\\ &= (A\tilde{v}, A\tilde{v}-\mu\tilde{v})+\mu\alpha^2(A\tilde{v}, A\tilde{u}-\mu\tilde{u})\\ &= |A\tilde{v}-\mu\tilde{v}|^2+\mu\alpha^2(A\tilde{v}-\mu\tilde{v}, A\tilde{u}-\mu\tilde{u})\\ &\quad +\mu(\tilde{v}, A\tilde{v}-\mu\tilde{v})+\mu^2\alpha^2(\tilde{v}, A\tilde{u}-\mu\tilde{u})\end{aligned}$$

and

$$\begin{aligned}(\tilde{v}, A\tilde{v}-\mu\tilde{v})+\mu\alpha^2(\tilde{v}, A\tilde{u}-\mu\tilde{u}) &= (\tilde{v}, A\tilde{v})-\mu(\tilde{v}, \tilde{v})+\mu\alpha^2(\tilde{v}, A\tilde{u})-\alpha^2\mu^2(\tilde{v}, \tilde{u})\\ &= \mu+\alpha^2\mu^2-\mu(\tilde{v}, \tilde{v})+\mu\alpha^2(\tilde{v}, A\tilde{u})-\alpha^2\mu^2\\ &= \mu(1-(\tilde{v}, \tilde{v})+\alpha^2(A\tilde{u}, \tilde{v}))\\ &= \mu(1-(\tilde{u}, \tilde{v}))=0\end{aligned}$$

Therefore, we have

$$(A\tilde{v}, A\tilde{v}-\|\tilde{v}\|^2\,\tilde{u}) = |A\tilde{v}-\mu\tilde{v}|^2+\mu\alpha^2[A\tilde{u}-\mu\tilde{u}]^2$$

Similarly,

$$\begin{aligned}(f/|u|, A\tilde{v}-\|\tilde{v}\|^2\,\tilde{u}) &= (f/|u|, A\tilde{v}-\mu\tilde{v})+(f/|u|, \mu\tilde{v}-\|\tilde{v}\|^2\,\tilde{u})\\ &= (f/|u|, A\tilde{v}-\mu\tilde{v})+\mu\alpha^2(f/|u|, A\tilde{u}-\mu\tilde{u})\\ &= (f/|u|, A\tilde{v}-\mu\tilde{v})\\ &\quad +\mu\alpha^2((I+\alpha^2A)^{-1/2}\,f/|u|, (I+\alpha^2A)^{1/2}\,(A\tilde{u}-\mu\tilde{u}))\\ &\leqslant \frac{|f|^2}{2\nu[u]^2}+\frac{\nu}{2}\,|A\tilde{v}-\mu\tilde{v}|^2\\ &\quad +\alpha^2\mu\left(\frac{|(1+\alpha^2A)^{-1/2}\,f|^2}{2\nu[u]^2}+\frac{\nu}{2}\,[A\tilde{u}-\mu\tilde{u}]^2\right)\\ &\leqslant (1+\alpha^2\mu)\,\frac{|f|^2}{2\nu[u]^2}+\frac{\nu}{2}\,(|A\tilde{v}-\mu\tilde{v}|^2+\mu\alpha^2[A\tilde{u}-\mu\tilde{u}]^2)\end{aligned}$$

Thus,

$$\frac{d}{dt}\,\|\tilde{v}\|^2+\nu(|A\tilde{v}-\mu\tilde{v}|^2+\mu\alpha^2[A\tilde{u}-\mu\tilde{u}]^2)\leqslant (1+\alpha^2\mu)\,\frac{|f|^2}{\nu[u]^2} \tag{3.4}$$

Since

$$|A\tilde{v}-\mu\tilde{v}|^2+\mu\alpha^2[A\tilde{u}-\mu\tilde{u}]^2 \geqslant (1+\alpha^2\mu)[A\tilde{u}-\mu\tilde{u}]^2 \\ \geqslant (1+\alpha^2\mu)\min_{n\in\mathbb{N}}|\mu-\lambda_n|^2$$

we obtain

$$\frac{d}{dt}\|\tilde{v}\|^2+\nu(1+\alpha^2\mu)\min_{n\in\mathbb{N}}|\mu-\lambda_n|^2 \leqslant (1+\alpha^2\mu)\frac{|f|^2}{\nu[u]^2} \tag{3.5}$$

Let us suppose that

$$\|\tilde{v}(t_0)\|^2=(\lambda_n+\kappa)(1+\alpha^2(\lambda_n+\kappa))$$

for some $t_0\in[0,T)$. In that case $\mu(t_0)=\lambda_n+\kappa$, so (3.5) implies

$$\frac{d}{dt}\|\tilde{v}(t_0)\|^2 \leqslant (1+\alpha^2\mu)\left(\frac{|f|^2}{\nu[u(t_0)]^2}-\nu\kappa^2\right)$$

The right-hand side is negative by one of our assumptions. Therefore, $\frac{d}{dt}\|\tilde{v}(t_0)\|^2<0$, so $\|\tilde{v}(t)\|$ decreases in a neighborhood of t_0. This proves the theorem. □

Corollary 1. *Let $0<\kappa<\lambda_1/2$, $u_0\in\mathcal{M}_{n,\kappa}$, and $v_0:=(I+\alpha^2A)\,u_0$. If $[u_0]>|f|/\nu\kappa$, then there exists a unique $t_0\geqslant 0$ such that*

$$[S(t)\,u_0]>\frac{|f|}{\nu\kappa}, \qquad t\in(-\infty,t_0)$$

$$[S(t)\,u_0]\leqslant\frac{|f|}{\nu\kappa}, \qquad t\in[t_0,\infty)$$

and

$$\frac{\|(I+\alpha^2A)\,S(t)\,u_0\|^2}{[S(t)\,u_0]^2}\leqslant(\lambda_n+\kappa)(1+\alpha^2(\lambda_n+\kappa)), \qquad t\in[0,t_0]$$

Proof. This follows from the Theorem 1, and the inequality (2.13). □

Corollary 2. *For each $n\in\mathbb{N}$, $0<\kappa<\lambda_1/2$,*

$$\mathcal{M}_{n,\kappa}\subset\left\{u_0\in V:[u_0]\leqslant\frac{|f|}{\nu\kappa}\right\}\cup\mathcal{C}_{n,\kappa}$$

Proof. This follows trivially from the last corollary. □

Theorem 2. *Let $u_0 \in \mathcal{G} \setminus \mathcal{A}$, and let $u(t) := S(t)\, u_0$, $v(t) := (I+\alpha^2 A)$ $S(t)\, u_0$. Then,*

$$\lim_{t \to -\infty} \frac{\|v(t)\|^2}{[u(t)]^2} \in \{\lambda_1(1+\alpha^2\lambda_1), \lambda_2(1+\alpha^2\lambda_2), \ldots\} \cup \{\infty\}$$

Therefore,

$$\mathcal{M}_{n,\kappa} = \mathcal{A} \cup \left\{ u_0 \in \mathcal{G} \setminus \mathcal{A} : \lim_{t \to -\infty} \frac{\|(I+\alpha^2 A)\, S(t)\, u_0\|^2}{[S(t)\, u_0]^2} \right.$$
$$\left. \in \{\lambda_1(1+\alpha^2\lambda_1), \lambda_2(1+\alpha^2\lambda_2), \ldots, \lambda_n(1+\alpha^2\lambda_n)\} \right\}$$
$$= \{u_0 \in \mathcal{G} : [S(t)\, u_0] = \mathcal{O}(e^{(1+\epsilon)\,\nu\lambda_n |t|}) \text{ as } t \to -\infty, \forall \epsilon > 0\}$$

Proof. If we rewrite (3.5) in its integral form, we can easily see that the limes

$$\lim_{t \to -\infty} \frac{\|v(t)\|^2}{[u(t)]^2} \in [0, \infty]$$

exists. Since $u_0 \notin \mathcal{A}$, $\lim_{t \to -\infty} [u(t)] = \infty$. Suppose that the above limes is finite. Then, $\lim_{t \to -\infty} \mu(t)$ is also finite. Therefore, by (3.5),

$$0 \leqslant \limsup_{t \to -\infty} \frac{d}{dt} \|\tilde{v}\|^2 \leqslant -\nu \liminf_{t \to -\infty} (1+\alpha^2\mu) \min_{n \in \mathbb{N}} |\mu - \lambda_n|^2$$
$$+ \limsup_{t \to -\infty} (1+\alpha^2\mu) \frac{|f|^2}{\nu[u]^2}$$
$$= -\nu \lim_{t \to -\infty} (1+\alpha^2\mu(t)) \min_{n \in \mathbb{N}} |\lim_{t \to -\infty} \mu(t) - \lambda_n|^2$$

since otherwise the limes would not exist. This proves the theorem. □

The previous theorem tells us that the set $\mathcal{M}_{n,\kappa}$ doesn't depend on the choice of $0 < \kappa < \lambda_1/2$. Therefore, we define $\mathcal{M}_n := \mathcal{M}_{n,\kappa}$ for any choice of $0 < \kappa < \lambda_1/2$.

Corollary 3. *For $u_0 \in \mathcal{M}_n$ such that $[u_0] > 4\,|f|/\nu\lambda_1$, we have*

$$\frac{\|(I+\alpha^2 A)\, u_0\|^2}{[u_0]^2} \leqslant \left(\lambda_n + \frac{2\,|f|}{\nu[u_0]}\right)\left(1+\alpha^2\left(\lambda_n + \frac{2\,|f|}{\nu[u_0]}\right)\right) \tag{3.6}$$

Proof. This follows trivially from Corollary 2 taking $\kappa = 2\,|f|/\nu[u_0]$. □

Lemma 1. *For $u \in A^{-1}V$, $v := (I+\alpha^2 A)\, u$, we have*

$$\frac{\|v\|^2}{|v|^2} \geqslant \frac{|[u]|^2}{[u]^2}$$

Also

$$\frac{\|v\|^2}{[u]^2} \leqslant \mu(1+\alpha^2\mu) \Rightarrow \frac{|[u]|^2}{[u]^2} \leqslant \mu$$

for $\mu > 0$.

Proof. Observe that

$$\begin{aligned}
\frac{\|v\|^2}{|v|^2} - \frac{|[u]|^2}{[u]^2} &= \frac{1}{|v|^2}\left((Av, v) - \frac{(v, v)}{(v, u)}\,(Av, u)\right)\\
&= \frac{1}{|v|^2}\left(Av,\, v - \frac{(v, v)}{(v, u)}\, u\right)\\
&= \frac{\alpha^2}{|v|^2}\left(Av,\, Au - \frac{(Av, u)}{(v, u)}\, u\right)\\
&= \frac{\alpha^2}{|v|^2}\left(Av - \frac{(Av, u)}{(v, u)}\, v,\; Au - \frac{(Av, u)}{(v, u)}\, u\right)\\
&= \frac{\alpha^2}{|v|^2}\left[Au - \frac{(Av, u)}{(v, u)}\, u\right]^2 \geqslant 0
\end{aligned}$$

Since

$$\|\tilde{v}\|^2 = \frac{\|v\|^2}{|v|^2}\,\frac{|v|^2}{[u]^2} = \frac{\|v\|^2}{|v|^2}\left(1+\alpha^2\,\frac{|[u]|^2}{[u]^2}\right)$$

the last statement follows. □

Remark 1. For the solution u of (VCHE) that satisfies the conditions in Theorem 1, we obtain the estimate

$$[u(t)]^2 \geqslant \left([u(0)]^2 + \frac{|f|^2}{8\nu^2(\lambda_n+\kappa)^2} \right) e^{-4\nu(\lambda_n+\kappa)\,t} - \frac{|f|^2}{8\nu^2(\lambda_n+\kappa)^2} \tag{3.7}$$

for $t \in [0, T]$.

Proof. For $t \in [0, T]$, we obtain using the last remark and Young's inequality

$$\begin{aligned}\frac{d}{dt}[u]^2 &= -2\nu(Av, u) + 2(f, u)\\ &\geqslant -2\nu(\lambda_n+\kappa)[u]^2 - 2\nu(\lambda_n+\kappa)\,|u|^2 - \frac{|f|^2}{2\nu(\lambda_n+\kappa)}\\ &\geqslant -4\nu(\lambda_n+\kappa)[u]^2 - \frac{|f|^2}{2\nu(\lambda_n+\kappa)}\end{aligned}$$

By the Gronwall inequality, we obtain then (3.7). □

The next result provides us with a method for producing elements of the sets $\mathscr{M}_n$. It will be used for all further results. But first, we need to state the following elementary fact:

Remark 2. If $\{u_k\}$ is a bounded sequence in V and $\lim_{k\to\infty} |u_k - u_0| = 0$ for some $u_0 \in H$, then $u_0 \in V$ and $\|u_0\| \leqslant \liminf_{k\to\infty} \|u_k\|$.

Theorem 3. *Let $u_1, u_2, \ldots \in V$, and let $t_1 > t_2 > \cdots$ be such that $\lim_{t\to-\infty} t_j = -\infty$. Suppose that for the initial data u_j the solution $S(t)\,u_j$ exists on the interval $[t_j, \infty)$. Let us also assume*

$$[u_k] \leqslant M, \qquad k \in \mathbb{N} \tag{3.8}$$

for some constant $M > 0$, and

$$[S(t_k)\,u_k] \geqslant \frac{|f|}{\nu\kappa}, \qquad k \in \mathbb{N} \tag{3.9}$$

Let there exist some $n \in \mathbb{N}$ such that

$$\frac{\|(I+\alpha^2 A)\,S(t_k)\,u_k\|^2}{[S(t_k)\,u_k]^2} \leqslant (\lambda_n+\kappa)(1+\alpha^2(\lambda_n+\kappa)), \qquad k \in \mathbb{N} \tag{3.10}$$

Then, there exist $u_\infty \in \mathcal{M}_n$ *and a subsequence* $\{u_{k_j}\}$ *of* $\{u_k\}$ *such that*

$$\lim_{j\to\infty} [S(t)\, u_{k_j} - S(t)\, u_\infty] = 0, \qquad t \in \mathbb{R} \tag{3.11}$$

Proof. We first want to prove that, for every $t \in \mathbb{R}$, there exists a constant $C(t) > 0$ such that

$$\|(I+\alpha^2 A)\, S(t)\, u_k\|^2 \leqslant C(t) \tag{3.12}$$

for k big enough. Without loss of generality, we may assume $M > |f|/\nu\lambda_1$. First, we fix $k \in \mathbb{N}$. By Lemma 1, there exists a unique $\beta_k \geqslant t_k$ such that

$$\begin{aligned} &[S(t)\, u_k] \geqslant \frac{|f|}{\nu\kappa}, \qquad t \in [t_k, \beta_k] \\ &[S(t)\, u_k] \leqslant \frac{|f|}{\nu\kappa}, \qquad t \geqslant \beta_k \end{aligned} \tag{3.13}$$

and

$$\frac{\|(I+\alpha^2 A)\, S(t)\, u_k\|^2}{[S(t)\, u_k]^2} \leqslant (\lambda_n + \kappa)(1+\alpha^2(\lambda_n+\kappa)), \qquad t \in [t_k, \beta_k] \tag{3.14}$$

Observe that by (2.11) and (3.22), we obtain

$$\begin{aligned} \frac{|f|^2}{\nu^2\kappa^2} &= [S(\beta_k)\, u_k]^2 \\ &\leqslant [u_k]^2\, e^{-\nu\lambda_1\beta_k} + \frac{|f|^2}{\nu^2\lambda_1^2}(1 - e^{-\nu\lambda_1\beta_k}) \\ &\leqslant e^{-\nu\lambda_1\beta_k}\left(M^2 - \frac{|f|^2}{\nu^2\lambda_1^2}\right) + \frac{|f|^2}{\nu^2\lambda_1^2} \end{aligned}$$

Therefore, we obtain an upper bound on β_k's

$$\beta_k \leqslant \frac{1}{\nu\lambda_1} \log \frac{\kappa^2(\nu^2\lambda_1^2 M^2 - |f|^2)}{(\lambda_1^2 - \kappa^2)\, |f|^2} =: t_M \tag{3.15}$$

With $C_1, C_2, \ldots$ being various constants, (3.7) implies now

$$
\begin{aligned}
[S(t)\, u_k]^2 &\leqslant \left([S(\beta_k)\, u_k]^2 + \frac{|f|^2}{8\nu^2\lambda_1^2} \right) e^{-4\nu\lambda_{n+1}(t-\beta_k)} \\
&= \left(\frac{|f|^2}{\nu^2\kappa^2} + \frac{|f|^2}{8\nu^2\lambda_1^2} \right) e^{-4\nu\lambda_{n+1}(t-\beta_k)} \\
&= C_1 e^{-4\nu\lambda_{n+1}(t-\beta_k)}, \qquad t \in [t_k, \beta_k]
\end{aligned}
$$

From here, using (3.14) and (3.15), we conclude

$$
\begin{aligned}
\|(I+\alpha^2 A)\, S(t)\, u_k\|^2 &\leqslant (\lambda_n + \kappa)(1 + \alpha^2(\lambda_n + \kappa))\, C_1 e^{-4\nu\lambda_{n+1}(t-\beta_k)} \\
&\leqslant C_2 e^{-4\nu\lambda_{n+1}(t-t_M)}, \qquad t \in [t_k, \beta_k]
\end{aligned} \tag{3.16}
$$

On the other hand, for $t \geqslant \beta_k$ from (2.13) and the previous calculations, we obtain

$$
\begin{aligned}
\|(I+\alpha^2 A)\, S(t)\, u_k\|^2 &\leqslant \|(I+\alpha^2 A)\, S(\beta_k)\, u_k\|^2\, e^{-\nu\lambda_1(t-\beta_k)} \\
&\quad + \frac{|f|^2}{\nu^2\lambda_1} (1 - e^{-\nu\lambda_1(t-\beta_k)}) \\
&\leqslant (\lambda_n + \kappa)(1 + \alpha^2(\lambda_n + \kappa))[S(\beta_k)]^2\, e^{-\nu\lambda_1(t-\beta_k)} \\
&\quad + \frac{|f|^2}{\nu^2\lambda_1} (1 - e^{-\nu\lambda_1(t-\beta_k)}) \\
&\leqslant (\lambda_n + \kappa)(1 + \alpha^2(\lambda_n + \kappa))\, \frac{|f|^2}{\nu^2\kappa^2} =: C_3
\end{aligned}
$$

This and (3.16) together imply

$$
\|(I+\alpha^2 A)\, S(t)\, u_k\|^2 \leqslant \max\{C_2 e^{-4\nu\lambda_{n+1}(t-t_M)}, C_3\} =: C(t), \qquad t \geqslant t_k \tag{3.17}
$$

Now we may use the Cantor diagonal process to extract a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that the limits

$$
\lim_{j \to \infty} S(t_{k_i})\, u_{k_j} =: w_i \in V, \qquad i \in \mathbb{N}
$$

exist in the space V. Since $S(t)\colon V \to V$ is a continuous mapping for $t \geqslant 0$, we obtain

$$
w_i = S(t_{k_i} - t_{k_j})\, w_j, \qquad j \leqslant i, \quad i, j \in \mathbb{N}
$$

Letting

$$u_\infty := S(-t_{k_1})\, w_1$$

we obtain

$$u_\infty = S(-t_{k_j})\, w_j, \qquad j \in \mathbb{N}$$

Therefore, $u_\infty \in \mathscr{G}$. Again, by the continuity of $S(t)\colon V \to V$, we obtain

$$\lim_{j \to \infty} [S(t)\, u_{k_j} - S(t)\, u_\infty] = 0, \qquad t \in \mathbb{R}$$

It remains to prove that $u_\infty \in \mathscr{M}_n$. To this end, we consider two cases. If $\liminf_{k \to \infty} \beta_k = -\infty$, then (3.11) and (3.13) imply

$$[S(t)\, u_\infty] \leqslant \frac{|f|}{\nu\kappa}, \qquad t \in \mathbb{R}$$

and, thus, $u_\infty \in \mathscr{A}$. If, on the other hand, $\liminf_{k \to \infty} \beta_k = \beta_\infty > -\infty$, Remark 2 together with (3.11) and (3.14), gives

$$\frac{\|(I + \alpha^2 A)\, S(t)\, u_\infty\|^2}{[S(t)\, u_\infty]^2} \leqslant (\lambda_n + \kappa)(1 + \alpha^2(\lambda_n + \kappa)), \qquad t \leqslant \beta_\infty$$

In both cases $u_\infty \in \mathscr{M}_n$. □

Theorem 4. *For each $n \in \mathbb{N}$, $\mathscr{M}_n$ is a connected, locally compact subset of V.*

Proof. Since $\mathscr{M}_n$ is a $S(\cdot)$-invariant set containing $\mathscr{A}$, it is connected as well. In order to prove the local compactness, it suffices to check that every sequence $u_1, u_2, \ldots \in \mathscr{M}_n \setminus \mathscr{A}$, which is bounded in V, has a subsequence converging to an element of $\mathscr{M}_n$. Since $\lim_{t \to -\infty} [S(t)\, u_k] = \infty$ for $k \in \mathbb{N}$, there exist $t_1 > t_2 > \cdots$ such that $\lim_{t \to -\infty} t_j = -\infty$ and $[S(t_k)\, u_k] \geqslant |f|/\lambda_1 \kappa$. Theorem 3 and Corollary 1 imply the wanted. □

4. RICHNESS OF THE SETS $\mathscr{M}_n$

One of the main results of this paper is the following theorem on the richness of the sets $\mathscr{M}_n$.

Theorem 5. *Let $n \in \mathbb{N}$. For every $p_0 \in P_n H$, there exists $u_\infty \in \mathscr{M}_n$ such that $P_n u_\infty = p_0$. In other words, $P_n H = P_n \mathscr{M}_n$.*

First, we need to prove a series of lemmas.

Lemma 2. *Let $u \in \mathscr{C}_{n,\kappa}$ for some $n \in \mathbb{N}$, and let $v := (I + \alpha^2 A)\, u$. Then*

$$[Q_n u]^2 \leqslant \frac{1}{\lambda_{n+1} - \lambda_n - \kappa} ((\lambda_n + \kappa)[P_n u]^2 - |[P_n u]|^2) \tag{4.1}$$

In particular

$$[Q_n u]^2 \leqslant \frac{\lambda_n + \lambda_{n+1}}{\lambda_{n+1} - \lambda_n} [P_n u]^2 \tag{4.2}$$

and

$$[u]^2 \leqslant \frac{2\lambda_{n+1}}{\lambda_{n+1} - \lambda_n} [P_n u]^2 \tag{4.3}$$

Proof. Observe that by Lemma 3, we have

$$\frac{|[u]|^2}{[u]^2} \leqslant \lambda_n + \kappa$$

Therefore,

$$\begin{aligned} [Q_n u]^2 &\leqslant \frac{1}{\lambda_{n+1}} |[Q_n u]|^2 \\ &= \frac{1}{\lambda_{n+1}} (|[u]|^2 - |[P_n u]|^2) \\ &\leqslant \frac{\lambda_n + \kappa}{\lambda_{n+1}} [Q_n u]^2 + \frac{1}{\lambda_{n+1}} ((\lambda_n + \kappa)[P_n u]^2 - |[P_n u]|^2) \end{aligned}$$

and (4.1) follows. The other two inequalities follow from (4.1) and the fact that $\kappa < \lambda_1 \leqslant \lambda_{n+1} - \lambda_n$ and both inequalities follow. □

Lemma 3. *Let $u \in \mathscr{M}_n$, for some $n \in \mathbb{N}$. Then, the following estimates hold:*

$$[Q_n u] \leqslant \max \left\{ \frac{|f|}{\nu\kappa}, \left(\frac{\lambda_n + \lambda_{n+1}}{\lambda_{n+1} - \lambda_n} \right)^{1/2} [P_n u] \right\} \tag{4.4}$$

and

$$|Q_n u| \leqslant \alpha_n + \beta_n [P_n u] \tag{4.5}$$

where

$$\alpha_n := \frac{2\,|f|}{\nu\lambda_1 (1+\alpha^2\lambda_{n+1})^{1/2}}$$

and

$$\beta_n := \left(\frac{2\lambda_{n+1}}{1+\alpha^2\lambda_{n+1}}\right)^{1/2} \cdot \frac{1}{(\lambda_{n+1}-\lambda_n)^{1/2}}$$

Also, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} \beta_n = 0$.

Proof. If $[u] \leqslant \frac{|f|}{\nu\kappa}$, then

$$[Q_n u] \leqslant [u] \leqslant \frac{|f|}{\nu\kappa}$$

If, on the other hand, $[u] \geqslant \frac{|f|}{\nu\kappa}$, we get

$$\|(I+\alpha^2 A)\, u\|^2 \leqslant (\lambda_n + \kappa)(1+\alpha^2(\lambda_n+\kappa))[u]^2$$

by Corollary 1. Applying Lemma 2, we obtain (4.4). Equation (4.5) follows from (4.4), and

$$|Q_n u| \leqslant (1+\alpha^2\lambda_{n+1})^{-1/2}\, [Q_n u]$$

By the properties (2.1) and (2.2) of the eigenvalues of the Stokes operator A, we obtain $\lim_{n\to\infty} \alpha_n = 0$ and $\liminf_{n\to\infty} \beta_n = 0$. □

Lemma 4. *Let $p_0 \in P_n H$ for some $n \in \mathbb{N}$. Then, for every $t_0 > 0$, there exists $w_0 \in P_n H$ such that $P_n S(t_0)\, w_0 = p_0$.*

Proof. First observe that, for every $u_0 \in P_n H$ such that $[u_0] > \frac{|f|}{\nu\kappa}$, there exists

$$\tau(u_0) = \min \left\{ \tau > 0 : [P_n S(\tau)\, u_0] = \frac{|f|}{\nu\kappa} \right\} > 0$$

Note also that with $v_0 := (I+\alpha^2 A)\, u_0$ we have

$$\frac{\|v_0\|^2}{[u_0]^2} \leqslant \lambda_n(1+\alpha^2\lambda_n) \leqslant (\lambda_n+\kappa)(1+\alpha^2(\lambda_n+\kappa))$$

Therefore, by Theorem 1, we have for $t \in [0, \tau_0]$

$$\frac{\|(I+\alpha^2 A)\, S(t)\, u_0\|^2}{[S(t)\, u_0]^2} \leqslant (\lambda_n+\kappa)(1+\alpha^2(\lambda_n+\kappa))$$

and

$$[S(t)\, u_0]^2 \geqslant [u_0]^2\, e^{-4\nu\lambda_{n+1}t} - \frac{|f|^2}{8\nu^2\lambda_n^2}$$

Lemma 2 now implies

$$[S(t)\, u_0]^2 \leqslant \frac{2\lambda_{n+1}}{\lambda_{n+1}-\lambda_n}\, [P_n S(t)\, u_0]^2$$

for $t \in [0, \tau_0]$. Therefore,

$$\begin{aligned} [P_n S(t)\, u_0]^2 &\geqslant \frac{\lambda_{n+1}-\lambda_n}{2\lambda_{n+1}}\, [S(t)\, u_0]^2 \\ &\geqslant \frac{\lambda_{n+1}-\lambda_n}{2\lambda_{n+1}} \left([u_0]^2\, e^{-4\nu\lambda_{n+1}t} - \frac{|f|^2}{8\nu^2\lambda_n^2} \right), \qquad t \in [0, \tau_0] \end{aligned} \tag{4.6}$$

For $r > 0$ we define $B(r) := \{u_0 \in V : [u_0] < r\}$ and $B_n(r) = B(r) \cap P_n H$. We now want to prove that there exists

$$r_0 > \frac{|f|}{\nu\kappa} \tag{4.7}$$

such that

$$[P_n S(t)\, u_0] > [p_0], \qquad u_0 \in P_n H \setminus B_n(r_0), \qquad t \in [0, t_0] \tag{4.8}$$

It is sufficient to consider only the case $[p_0] \geqslant \frac{|f|}{\nu\kappa}$. We fix any $r_0 > \frac{|f|}{\nu\kappa}$ such that

$$\frac{\lambda_{n+1}-\lambda_n}{2\lambda_{n+1}} \left(r_0^2 e^{-4\nu\lambda_{n+1}t_0} - \frac{|f|^2}{8\nu^2\lambda_n^2} \right) > [p_0]^2 \tag{4.9}$$

For $[u_0] \geqslant r_0$, we obtain

$$[p_0]^2 \geqslant \left(\frac{|f|}{\nu\kappa}\right)^2 = [P_n S(\tau_0)\, u_0]^2$$
$$\geqslant \frac{\lambda_{n+1}-\lambda_n}{2\lambda_{n+1}}\left(r_0^2 e^{-4\nu\lambda_{n+1}\tau_0} - \frac{|f|^2}{8\nu^2\lambda_n^2}\right) \tag{4.10}$$

Combining (4.9) and (4.10), we get $t_0 < \tau_0$. In particular, (4.6) holds for $t \in [0, t_0]$. From (4.6) and (4.9) we obtain

$$[P_n S(t)\, u_0] > [p_0] \tag{4.11}$$

for all $u_0 \in P_n H \setminus B_n(r_0)$, $t \in [0, t_0]$. In order to prove the lemma, let us choose a continuous function $\theta\colon \mathbb{R} \to \mathbb{R}$ such that $\theta(x) = 1$ for $x \leqslant r_0$ and $\theta(x) = 0$ for $x \geqslant 2r_0$. Define

$$g(u_0) = P_n S(\theta([u_0]\, t_0)\, u_0, \qquad u_0 \in P_n H$$

By (2.11) we have

$$S(t)\, B(r) \subset B(r), \qquad r \geqslant \frac{|f|}{\nu\lambda_1}, \qquad t \geqslant 0$$

Therefore,

$$g(B_n(2r_0)) \subset B_n(2r_0)$$

Also, g is continuous, and it satisfies $g(u_0) = u_0$ for $[u_0] = 2r_0$. We can choose r_0 large enough that $p_0 \in B_n(2r_0)$. By Brouwer's Fixed Point Theorem, there exists $w_0 \in B_n(2r_0)$ such that $g(w_0) = p_0$. If we show that $[w_0] \leqslant r_0$, by the definition of g, we would have

$$P_n S(t_0)\, w_0 = g(w_0) = p_0$$

and this is exactly the claim of this lemma. To this end, let's assume $[w_0] > r_0$. Then,

$$[g(w_0)] = [P_n S(\theta([w_0]\, t_0)\, w_0] > [p_0]$$

by (4.8). This is a contradiction, and the lemma is proven. □

Now we have the tools to prove Theorem 5.

Proof of Theorem 5. Let $p_0 \in P_n H$. Choose a sequence $0 > t_1 > t_2 > t_3 > \cdots$ such that $\lim_{k\to\infty} t_k = -\infty$. By Lemma 4, there exist $u_1, u_2, \ldots \in V$ and $p_1, p_2, \ldots \in P_n H$ such that

$$S(-t_k)\, p_k = u_k, \qquad k \in \mathbb{N}$$

and

$$P_n u_k = p_0, \qquad k \in \mathbb{N}$$

Let us define

$$U_k(t) := S(t)\, u_k, \qquad t \geqslant t_k$$

and

$$V_k(t) := (I + \alpha^2 A)\, U_k(t), \qquad t \geqslant t_k$$

Let us assume first that

$$[p_k] \leqslant \frac{|f|}{\nu\kappa} \tag{4.12}$$

for infinitely many $k \in \mathbb{N}$. Without loss of generality, we may assume that this is true for all $k \in \mathbb{N}$. By the Poincaré inequality, we have

$$\|V_k(t_k)\| = \|(1+\alpha^2 A)\, p_k\| \leqslant \lambda_n^{1/2}(1+\alpha^2\lambda_n)^{1/2}\, [p_k] \leqslant \lambda_n^{1/2}(1+\alpha^2\lambda_n)^{1/2}\, \frac{|f|}{\nu\kappa}$$

The inequality (3.21) gives

$$\|V_k(t)\| \leqslant \lambda_n^{1/2}(1+\alpha^2\lambda_n)^{1/2}\, \frac{|f|}{\nu\kappa}, \qquad t \geqslant t_k$$

Therefore,

$$[U_k(t)] \leqslant \frac{\lambda_n^{1/2}(1+\alpha^2\lambda_n)^{1/2}}{\lambda_1^{1/2}(1+\alpha^2\lambda_1)^{1/2}}\, \frac{|f|}{\nu\kappa}, \qquad t \geqslant t_k \tag{4.13}$$

Similarly, as in the Theorem 3, we may, by passing to a subsequence, assume that

$$\lim_{k\to\infty} [U_k(t) - S(t)\, u_\infty] = 0, \qquad t \in \mathbb{R}$$

for some $u_\infty \in \mathcal{G}$. From here it follows that

$$P_n u_\infty = P_n u_1 = p_0$$

and, using (4.13), $u_\infty \in \mathcal{A} \subset \mathcal{M}_n$. This proves the Lemma under the assumption (4.12). Let us now assume that

$$[p_k] \geqslant \frac{|f|}{\nu\kappa}$$

for infinitely many $k \in \mathbb{N}$. Again, by passing to a subsequence, we may assume that this is true for all $k \in \mathbb{N}$. Since either $[u_k] \leqslant \frac{|f|}{\nu\kappa}$ or $[u_k] > \frac{|f|}{\nu\kappa}$ for each $k \in \mathbb{N}$, by Lemma 2

$$[u_k] \leqslant \max\left\{\frac{|f|}{\nu\kappa}, \left(\frac{2\lambda_{n+1}}{\lambda_{n+1}-\lambda_n}\right)[p_0]\right\}, \qquad k \in \mathbb{N}$$

Since

$$\frac{\|V_k(t_k)\|^2}{[U_k(t_k)]^2} = \frac{\|(I+\alpha^2 A)\, p_k\|^2}{[p_k)]^2} \leqslant (\lambda_n+\kappa)(1+\alpha^2(\lambda_n+\kappa)), \qquad k \in \mathbb{N}$$

the assumptions of the Theorem 3 are satisfied. We obtain $u_\infty \in \mathcal{M}_n$, so

$$\lim_{k \to \infty} [U_k(t) - S(t)\, u_\infty] = 0, \qquad t \in \mathbb{R}$$

Exactly as in the previous case we obtain $P_n u_\infty = p_0$. This completes the proof of the theorem. □

5. SOME DENSITY PROPERTIES OF THE SETS $\mathcal{M}_n$

This section is concerned with questions on the density of the sets $\mathcal{M}_n$. It consists of two results. The first one is an analogue of a result by Constantin *et al.* (1997), which positively answers the conjecture by Bardos and Tartar (1973). It concerns the (NSE), and postulates that $S(t)\, H$ is dense in H for $t > 0$. However, the density is shown in a weaker norm then the natural energy norm of (NSE). Here, we find ourselves in a similar situation. We prove that $\bigcup_{n \in \mathbb{N}} \mathcal{M}_n$ is dense in H, but again in a weaker norm then the natural energy norm for the (VCHE).

In the second result we show that the eigenvectors of the Stokes operator A corresponding to λ_n for some $n \in \mathbb{N}$ belong to the closure of scaled set $\mathcal{M}_n$. This time, the closure is taken with respect to the "natural" energy norm for the (VCHE). The analogue of this result for the (NSE) has been proved by C. Foias and I. Kukavica, and is yet to be published.

Theorem 6. *The set* $\bigcup_{n\in\mathbb{N}} \mathcal{M}_n (\subset \mathcal{G})$ *is dense in* H.

Proof. Let $u_0 \in H$ be arbitrary. By Theorem 3, for each $n \in \mathbb{N}$ there exists $u_n \in \mathcal{M}_n$ such that $P_n u_n = P_n u_0$. From Lemma 3, we have the estimate

$$|Q_n u_n| \leqslant \alpha_n + \beta_n [P_n u_n]$$

This implies

$$\begin{aligned}|u_n - u_0| &= |Q_n(u_n - u_0)| \leqslant |Q_n u_n| + |Q_n u_0| \\ &\leqslant \alpha_n + \beta_n [P_n u_n] + |Q_n u_0| = \alpha_n + \beta_n [P_n u_0] + |Q_n u_0|\end{aligned}$$

By Lemma 3, we obtain

$$\liminf_{n\to\infty} |u_n - u_0| = 0$$

and the theorem is proven. □

Theorem 7. *Let* w *be an eigenvector of the Stokes operator* A *corresponding to the eigenvalue* λ_n *for some* $n \in \mathbb{N}$, *i.e., let* $Aw = \lambda_n w$, *and let* $[w] = 1$. *Then*

$$w \in \overline{\left\{ \frac{u}{[u]} : u \in \mathcal{M}_n \right\}}$$

where the closure is taken in the norm $[\,\cdot\,]$. *In particular,*

$$P_n H \subset \overline{\underbrace{\mathcal{M}_n + \mathcal{M}_n + \cdots + \mathcal{M}_n}_{k_n}}$$

where $k_n := \dim P_n H = m_1 + m_2 + \cdots + m_n$, m_j *being the multiplicity of* λ_j.

Proof. Let $r > 4\,|f|/\nu\lambda_1$. By Theorem 6, the exists $u_r \in \mathcal{M}_n$, such that $P_n u_r = rw$. Also, by Lemma 2, and Corollary 3, we have

$$[Q_n u_r]^2 \leqslant \frac{\dfrac{2\,|f|}{\nu[u_r]}[P_n u_r]^2}{\lambda_{n+1} - \lambda_n - \dfrac{2\,|f|}{\nu[u_r]}} \leqslant \frac{2\,|f|\, r^2}{\nu r(\lambda_{n+1} - \lambda_n) - 2\,|f|}$$

Thus,

$$\frac{[Q_n u_r]^2}{r^2} \leqslant \frac{2\,|f|}{\nu r(\lambda_{n+1}-\lambda_n)-2\,|f|} \to 0$$

and

$$\frac{[u_r]^2}{r^2} = \frac{[P_n u_r]^2+[Q_n u_r]^2}{r^2} \to 1$$

when we let $r \to \infty$. Finally

$$\left[\frac{u_r}{[u_r]} - w\right] = \left[\frac{u_r - rw}{[u_r]} - w\left(1 - \frac{r}{[u_r]}\right)\right]$$
$$\leqslant \frac{[Q_n u_r]}{r} + \left|1 - \frac{r}{[u_r]}\right| \to 0$$

when $r \to \infty$. This proves the theorem. □

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