Review of

"Natural Linear Additive Functionals of Superprocesses"
"Nonlinear parabolic P.D.E. and Additive Functionals of Superdiffusions"
"Solutions of Nonlinear Differential Equations on a Riemannian Manifold and Their Trace on the Martin Boundary"

These three papers are part of a long effort by the authors and others to extend some of the most fruitful results linking Brownian motion to solutions of Laplace's or Poisson's equation to non-linear analogs that arise in the study of the family of superprocesses. These three papers extend many previously observed relationships to far more general settings. In summary, the authors identify conditions that are necessary for solutions to exist, and conditions which are sufficient. The key tool for the three papers is a new construction of additive functionals of superprocesses. These additive functionals yield probabilistic representations of solutions. These papers come amidst a large stream of related works primarily but not solely by the authors. Although they are not the final word on the subject, they are of great interest in their own right.

The story for the linear equations is familiar. Let L be a second order elliptic operator. If E is a bounded smooth domain and f is a continuous function in E and ϕ a bounded continuous function on ∂E then the solution of the boundary value problem $Lh = -f$ in E, $h = \phi$ on ∂E can be expressed classically in terms of the Green's function and Poisson kernel by $h(x) = \int_E g(x,y)f(y) + \int_{\partial E} k(x,y)\phi(y)\sigma(dy)$. A generalization to the boundary value problem with measures is the equation $Lh = -\eta$ in E and $h = \nu$ on ∂E. A mild solution is then given by the formula $h(x) = \int_E g(x,y)\eta(dy) + \int_{\partial E} k(x,y)\nu(dy) = G\eta + K\nu$.

Pioneering work on the connections between these representations and probabilistic representations was done by Kakutani, Kac, Doob and Hunt starting some 50 years ago. Probabilistically, if ξ is a diffusion with generator L and τ is its exit time from E a regular domain, then a solution to the first equation is also available through the following formula involving the expectation of the process started from x: $h(x) = E_x(\int_0^\tau f(\xi_s) + \phi(\xi(\tau)))$.

When $f = 0$ solutions are called L-harmonic and in this case $h = K\nu$. This illustrates that L-harmonic functions are determined by measures ν which are concentrated on the boundary of E. In fact, the positive solutions to $Lh = 0$ in an arbitrary domain $E \subset R^d$ are in a one-to-one correspondence with finite measures on the minimal part of the Martin boundary of E.

Viewed more generally, if $f(x) \geq 0$ then this represents the solution as $h(x) = P_x(A(0,\infty))$, where A is the additive functional of the process ξ given by $1_{[0,\tau)}f(\xi_s)ds + 1_\tau\phi(\xi_\tau)$ This relationship between additive functionals of a process and solutions to a related differential equations is extended in these works.

These papers deal with the following non-linear analogs to the linear equations above. Describe all solutions to the elliptic PDE in E, $L(u) - u^\alpha = 0$, and the parabolic PDE with measure data

$$\frac{\partial u}{\partial r} + Lu - u^\alpha = -\eta \text{ in } R_+ \times E$$
$$u = \nu \text{ on } (R_+ \times E)^e$$
$$u = 0 \text{ on } \{\infty\} \times E.$$

(Here the superscript e refers to the extremal part of the Martin boundary for the L-diffusion.) It is assumed in these papers that $1 < \alpha \leq 2$. Note that the non-linear problems need not have a unique solution.

Whereas previously Brownian motion, or more generally an L-diffusion is the correct probabilistic tool to represent solutions, in the non-linear case the tool used is the superprocess based on L with branching related to α. In a time-inhomogeneous setting, this is a set-indexed, measure-valued process, denoted X_U, where U is a space-time subset of $S = R_+ \times E$. Let μ be a finite measure on S, $1 < \alpha \leq 2$ and $(\xi_t, \Pi_{r,x})$ be an L-diffusion in E. Then the superprocess can be defined as a measure-valued Markov process with Laplace functional given by $P_\mu(\exp(-\langle f, X_U\rangle)) = \exp(-\langle u, \mu\rangle)$, where u solves the integral equation $u + \mathcal{E}_U(u) = K_U(u)$. In this setting, $\mathcal{E}_U(u)(r,x) = G_U(u^\alpha)(r,x) = \Pi_{r,x}(\int_r^\infty u(s,\zeta_s)^\alpha ds)$ and $K_U(u)(r,x) = \Pi_{r,x}(u(\tau,\zeta_\tau), \tau < \zeta)$, but this can change depending on context. More generally, the integral equation $u + \mathcal{E}(u) = h$ is termed the $\mathcal{E}$-equation and plays an essential role in these works. As one can see, under some conditions, a function u satisfying an $\mathcal{E}$-equation will solve a non-linear equation of the type above. If $U = \{t\} \times E$ then we abbreviate $X_U = X_t$, and if $U = [0,\infty) \times D$ then $X_U = X_D$. The latter are called the exit measures as they capture information about the particles as they first exit the domain D. They are related to the elliptic PDE.

The main results of the three papers may be summarized briefly as follows. An additive functional of the superprocess is used to represent solutions to the above equations. Let $h = K\nu$ (or $K\nu + G\eta$ in the parabolic case), then conditions are found, so that an additive functional A can be constructed with potential $P.(A) = h(\cdot)$, and log-potential $u(\cdot) = -\log P.(\exp(-A))$. The functions h and u are related by the $\mathcal{E}$-equation.

For the parabolic PDE, u is the desired solution. For the elliptic problem let Γ be a closed subset of the boundary, and let ν be a radon measure on $E' \setminus \Gamma$ (E' is the Martin boundary of E) satisfying some conditions. If $h = K\nu$, then for nice enough ν, $A = A_\nu$ is defined such that $u(x) = -\log P_x(\mathcal{R} \cap \Gamma = \emptyset, \exp(-A_\nu))$ solves $Lu = u^\alpha$ in E. This solution is termed the (Γ,ν)-solution. It need not solve the $\mathcal{E}$-equation for h.

Two important examples are if $\Gamma = \partial E$ and $\nu = 0$, then the solution $u_\Gamma(x) = -\log P_x(\mathcal{R} \cap \Gamma = \emptyset)$ is the maximal solution in E. Whereas, if the surface measure σ is defined on the boundary of E and $\nu = f\sigma$ where f is bounded and continuous and $\Gamma = \emptyset$, then $A_\nu = \langle X^E, f\rangle$, and the solution $u(x) = -\log P_x(\exp(-\langle X^E, f\rangle))$ has boundary value f.

Before stating in more detail the exact accomplishments of these papers, a brief digression into the relationship between superprocesses and non-linear differential equations is in order. Several authors have used the PDE that occurs in the log-Laplace transform to derive properties of the superprocess. Iscoe in 1986 introduced the use of an additive functional of the superprocess in finding solutions to the equation. His functionals are of the form $A_t = \langle X_t, f\rangle + \int_0^t \langle X_s, \phi\rangle ds$. In particular, he showed that $u(x) = -\log P_x(\exp(-A_t))$ solves $\dot{u} + Lu - u^\alpha = -\phi$, $u(0) = f$. Le Gall first characterized all solutions to $\Delta u = u^2$ in the unit circle in R^2 and later for $\mathcal{C}^2$ domains in R^2 in terms of a subset of the boundary and a radon measure on the boundary. He used the Brownian snake instead of the closely related superprocess. He found that solutions were in a one-to-one correspondence to certain pairs (Γ, ν), as above The pair (Γ, ν) is called a trace for a solution. The unique characterization of a solution by a trace doesn't quite carry over to the general case. Dimension 2 is special for the Brownian superprocess as points on the boundary of a smooth enough domain are "hit" by the exit measures. This means the boundary polar sets for the range of the superprocess are trivial. (Roughly, boundary polar means that the set of points on the boundary are not hit by exiting particles). He also noted that in higher dimensions the characterization couldn't be one-to-one. The analysts Gmira, Marcus and Veron have investigated solutions to this type of non-linear differential equations in the last decade. Their work deals with the more general case of $\alpha \geq 1$. Here too, a description of solutions in terms of a trace is given although its definition is no longer probabilistic.

The main results of all three papers build upon many previous papers by the authors. Two classes of sets are involved in most of the theorems. The authors call a subset of E' weakly exceptional if it has Martin capacity 0 (or Green capacity 0 if it is a subset of E), and strongly exceptional if it is polar for the range of the superprocess. (Polar for the range means roughly that the particles do not hit this set of points.) The author's previously have proved that strongly exceptional implies weakly and that for smooth enough domains the two classes are equivalent. Most of the key theorems of these papers are such that sufficient conditions to ensure that a solution exist are related to the measures involved not charging weakly exceptional sets, and necessity is related to measures not charging strongly exceptional sets. Note that the condition that a measure not charge weakly exceptional is a "stronger" (more restrictive) condition than that it not charge strongly exceptional sets.

Previously in "*Linear additive functionals of superdiffusions and related nonlinear P.D.E.*" the related problem of finding solutions in a bounded, smooth domain D to the equation $Lv - v^\alpha = -\eta$ in D and $v = \nu$ on ∂D was considered. They showed that if $h = G\eta + K\nu$ and η and ν did not charge weakly exceptional sets (suitably defined) then a solution to the $\mathcal{E}$-equation exists. In contrast, a necessary condition for a solution v for a given η and ν is that these measures not charge strongly exceptional sets. To establish these results, they constructed a class of linear additive functionals, generically denoted by an A, whose potential was h $(P_x(A_\infty) = h(x))$ and whose log potential $(u(x) = -\log P_x(\exp(-A_\infty)))$ is a solution to the $\mathcal{E}$-equation.

It took two papers, "*Natural Linear Additive Functionals of Superprocesses*" and "*Non-linear parabolic P.D.E. and Additive Functionals of Superdiffusions*", to extend these results from smooth domains to arbitrary domains in R^d. To do so, it required a new construction of their additive functionals. This is done in "*Natural Linear Additive Functionals of Superprocesses*". In this paper, natural (A_{t+} is predictable) linear additive functionals (NLA) are defined and constructed. Again, a NLA has its potential h and its log-potential u. It is shown that h and u are related by the $\mathcal{E}$-equation and that h is always an exit rule for the underlying L-diffusion. In converse, h being an exit rule satisfying the condition $h+\mathcal{E}(h)<\infty$ suffices to show h is potential for some NLA A. A limit of discrete functionals defined in terms of h gives A.

These results are applied and extended in *"Nonlinear parabolic P.D.E. and Additive Functionals of Superdiffusions"* to solving the non-linear equations in the time-homogeneous and inhomogeneous cases. Let E be an arbitrary domain in R^d, and η and ν be the measures in the parabolic boundary value problem described previously. Again, set $h = G\eta + K\nu$. Then, if a solution to $u + \mathcal{E}(u) = h$ exists it can be represented as the log-potential of some NLA A whose potential is h. In this case, the probabilistic representation always applies when a solution exists. Sufficient conditions are given for existence in terms of the spectral measure of h not charging weakly exceptional sets. Necessary conditions for existence are given in terms of these measures not charging strongly exceptional sets. When A exists it is shown that A splits into two pieces, $A_\eta + A_\nu$, the former determined by the events inside the domain, and the latter by events on the boundary. This extends the Iscoe formula. A key result is the proof that three seemingly different classes of functions for h candidates are in fact identical.

In the previous paper *"Solutions of $Lu = u^\alpha$ dominated by L-harmonic functions"* the authors investigated all solutions u to the elliptic equation $Lu = u^\alpha$ in E which are bounded by a harmonic function in E. These solutions are called moderate. They show for smooth domains that if h is a harmonic function with Martin representation $h = K\nu$ then the $\mathcal{E}$-equation for h has a solution if ν does not charge weakly exceptional sets. In contrast, if u is a moderate solution, and h its minimal harmonic majorant, then the ν appearing in the Martin representation for h does not charge strongly exceptional sets.

However, solutions to the elliptic PDE considered may be more singular than a harmonic function. A typical example is the maximal solution u_Γ defined previously. For E smooth and the process being super Brownian motion, this solution blows up like one over the distance to the boundary squared, whereas any harmonic function blows up at most like one over the distance.

In the paper *"Solutions of Nonlinear Differential Equations on a Riemannian Manifold and Their Trace on the Martin Boundary"* E is again allowed to be any Riemannian manifold. For each solution u to the elliptic PDE, a pair (Γ, ν) is defined. Again this pair is called the trace of u. Here Γ is a closed subset or E', and ν a radon measure on $E' \setminus \Gamma$ as described previously. One may think roughly that Γ is where the solution blows up badly and ν corresponds to the "moderate" part of the solution. The main results characterize which pairs are possible traces. The statements are similar to the ones for moderate solutions, but the domains considered are more general. These results hold for domains which have their Martin boundary coincide with their Euclidean boundary (actually in slightly more generality). It is shown that for a pair (Γ, ν) to be a trace of some solution it is sufficient that ν has some explosion property and it does not charge weakly exceptional sets. Conversely, if (Γ, ν) is the trace of some solution, then ν must have the explosion property with respect to Γ and not charge strongly exceptional sets.

In terms of the solutions they show if (Γ, ν) is a permissible trace, then the (Γ, ν)-

solution is indeed a solution, It will be moderate if and only if Γ is empty. Recall the (Γ, ν)-solution was defined in terms of an additive functional A_ν. The existence of this for the class of ν considered is a consequence of the work in "*Natural Linear Additive Functionals of Superprocesses*". The (Γ, ν)-solution will have trace (Γ, ν). If $\Gamma = \emptyset$ then the (Γ, ν) solution corresponds to a unique solution. However, for an arbitrary solution u, although its trace (Γ, ν) is uniquely defined, the corresponding (Γ, ν)-solution need not be u. It is always the maximal solution with this trace.

This non-uniqueness is addressed in a later paper "*Fine Topology and Fine Trace on the Boundary Associated with a Class of Semilinear Differential Equations*, where a new topology is introduced on the boundary of E an arbitrary domain in R^d (or even a Riemannian manifold). This is the fine topology, which is more closely linked to the singularities of solutions. Here, uniqueness of the trace is shown for solutions termed σ-moderate (increasing limits of moderate functions). It is unclear if all solutions are such, but no non-σ-moderate solutions are known. It should be noted that this paper does not use the superprocess but instead a conditioned L-diffusion and is valid for all $\alpha > 1$.

These papers are just three of several in the authors systematic attack on this important problem. Much progress is recorded here in pushing previous results to more general domains, but these are not the final word. For the parabolic problem, the gap between the strong and weak exceptional sets remains. For the elliptic problem, as indicated, in the more recent paper "*Fine Topology and Fine Trace on the Boundary Associated with a Class of Semilinear Differential Equations* a new topology on the boundary of the domain allows one to consider arbitrary domains and get unique probabilistic representations. However, there is still work to do as the gap between σ-moderate solutions and arbitrary solutions remains. Despite this, these papers make interesting reading. They are written in the authors characteristic style: clearly structured, well motivated and with a generous amount of valuable background material. The authors have worked quite hard to create the proper framework for these questions and in doing so they have created a powerful set of tools for approaching these problems. These include a nice mix of diffusion theory, additive functional theory and partial differential equations theory. For the reader, be prepared to spend some time looking at references to other papers by the authors when following the proofs. Many theorems appear similar to previous ones as the differences are technical. Typically only the level of generality has changed. It is a worthwhile endeavor though. For the interested, I would suggest the elliptic problem first as it seems more self contained.

Additional references:

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