The test covers material in chapter 4 and chapter 5 that we covered in class.

4.1 Continuous RVs.

1. What is a density, give an example
2. How does a density answer a question about probability?

Exercise 1

$$f(x) = \begin{cases} c(1-x^2) & 0 < x < 1 \\ 0 & o/w \end{cases}$$

1. *Find c*

   1 $c = 3/2$, *as* $\int_0^1 (1-x^2)dx = 2/3$

2. *Find* $P(1/2 < X)$

   2 $\int_{1/2}^2 f(x)dx = 5/16$

3. *Find* $F(x) = P(X < x)$

   3 $\int_0^a f(x)dx = -(a^3 - 3*a)/2$ *where* $0 < a < 1$.

4. *Find x so that* $F(x) = 1/2$.

   4 *Solve* $-(a^3 - 3*a)/2 = 0$ *at* $\sqrt(3)$

Expectation of X

1. What does the term expectation of X mean?
2. Why might one say the expectation is the center?

Exercise 2

1. *Find* $E(X)$

   1 $E(X) = \int xf(x)dx = 3/8$

2. *Find* $Var(x)$.

   2 $E(X^2) = \int x^2 f(x) = 1/5$, *so* $Var(x) = 1/5 - (3/8)^2 = 19/320$

The uniform

1. Why is the uniform called *uniform*?
2. Suppose a traffic light stays red for 90 seconds. If you drive up to a red light, explain why one might argue that the time you have to wait is uniform. What parameters?

Exercise 3

1. *Let X be uniform on* $[0,1]$. *Explain why the variance is* `more` *than the last answer.*

1 Data has same range but more spread out.

2. If U is uniform on $[10, 100]$ find $F(x)$

2 $\int_{10}^{a} 1/(100-10)dx = (a-10)/90$ *for* $10 < a < 100$.

3. Use the above to find the 25th percentile

3 Solve $F(a) = .25$*:* $a = 65/2$

Exponential

1. Why is this distribution given its name
2. What is the memoryless property?
3. WHy is the integral for Γ so useful here?

Exercise 4 *1. Let X be exponential with rate 10. Find the mean and median.*

1 mean is 1/10; median solves $F(a) = .5$ *or easier* $1 - F(a) = 0.5 = e^{-a/10}$. *So* $a = 10\log(2)$.

2. Let X be exponential with rate 100, find $P(X > 100)$

2 Just use $P(X > a) = e^{-a/\theta}$ *to get* e^{-1}.

Gamma Exercise 5 *Use the gamma integral to find $E(X^4)$ when X is exponential with rate 1.*

5 $E(X^4) = \int_0^\infty x^4 e^{-x} dx = \int_0^\infty x5 - 1 e^{-x} dx = \Gamma(5)$

Normal

1. Why is the normal normal?
2. How can you identify σ from the graph of the density?
3. Know the rules of thumb.

Exercise 6 *1. Use table to find* $P(-.5 < Z < .75)$.

1 I'm lazy using computer function `pnorm` here:

```
> pnorm(0.75) - pnorm(-0.5)
[1] 0.4648351
```

2. Which is greater $P(Z > 1)$ *or* $P(-1 < Z < 0)$*?*

2
```
> (1 - pnorm(1)) > (pnorm(0) - pnorm(-1))
[1] FALSE
```

3. If X is normal with mean 270 and sd 7 find $P(260 < X < 275)$.

3 pnorm((275 - 270)/7) - pnorm((260 - 270)/7)

[1] 0.685911

4. *For X above, find the 75th percentile*

*4 270 + 7 * qnorm(0.75)*

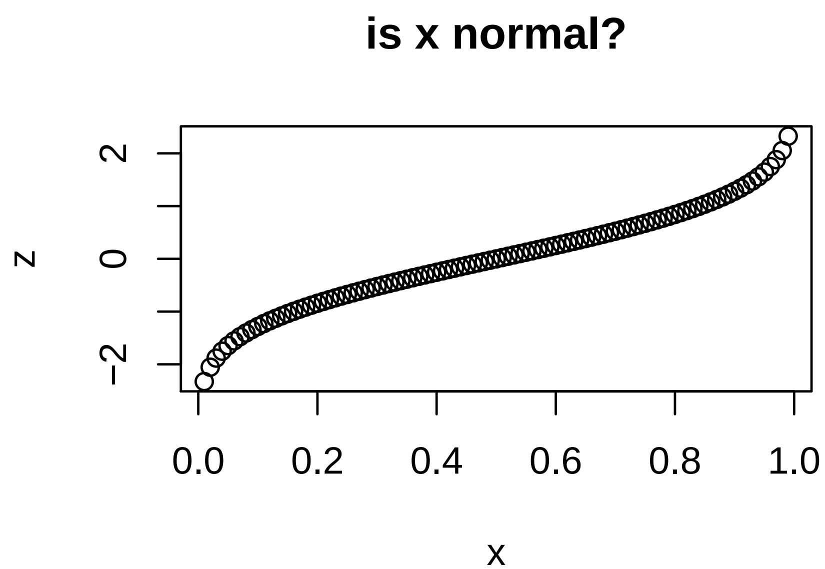
[1] 274.7214

5. *Find $E(X^4)$ using the gamma integral and substitution.*

5 *Write as $2\int_0^\infty x^4 f(x)dx$, change variable and get 3.*
(Sage commands: see
texttttsagenb.org)

```
f(x) = (1/sqrt(2*pi)) * e^(- x^2/2)
2*integrate(x^4 * f(x), x, 0, infinity)
```

6. *The quantile plot is of some distribution against the normal. Can you tell if the distribution is normal too?*



6 *No, the points are not on a straight line.*

Joint - discrete 1. Know how to relate a table of numbers to a probability question, like $P(X = x, Y = y)$.

Exercise 7 *1. Let the joint distribution of X (on row) and Y (column) be given by the following:*

```
> m
    Y
X          -1      0      1
  1 0.0667 0.2667 0.0667
  2 0.1333 0.0667 0.1333
  3 0.0667 0.1333 0.0667
```

Find the following:

(a) The marginal distribution of X

1a
```
> apply(m, 1, sum)
     1     2     3
0.400 0.333 0.267
```

(b) The marginal distribution of Y.

1b
```
> apply(m, 2, sum)
   -1     0     1
0.267 0.467 0.267
```

(c) The probability $P(X \leq 2, Y \geq 0))$

1c *Add the correct terms together:*

```
> sum(m[1:2, 2:3])
[1] 0.533
```

(d) The conditional probability $P(X \leq 2 \mid Y = 1)$

1d *This is the reweighted probability over the $Y = 1$ column*

```
> sum(m[1:2, 3])/sum(m[, 3])
[1] 0.75
```

Joint Continuous 1. The issue here is setting up the integrals!!!

1. Let $f(x) = c(2-(x+y))$ over the unit square: $0 < x, y < 1$. Find c

   1 c solves $1/c = \int_0^1 \int_0^1 f(x) dy dx = 1$ so $c = 1$

2. (U, V) is a randomly chosen point over the unit circle $x^2 + y^2 < 1$. Find the joint density:

   2 The density is $f(x, y) = c$ when $x^2 + y^2 < 1$, Since the volume must be 1 and the volume is a cylinder with base 1 and height c, $c = 1$.

3. Pick U and V uniform on the interval $[0, 1]$. Find the probability that $|U - V| < 1/2$.

3 This is like the problem in class. The joint density is just $c = 1$. So we can set this up as an integral:

$$\int_{x=0}^{1} \int_{y=x-1/2}^{x+1/2} f(x,y)dydx$$

But this is a bit tricky $f(x,y)$ is 0 on some of this. We can break into two integrals:

$$\int_{x=0}^{1/2} \int_{0}^{x+1/2} dydx + \int_{x=1/2}^{1} \int_{y=x-1/2}^{1} dydx$$

Both are 3/8 so the answer is 6/8 = 3/4.

Here we simulate to check

```
> x <- runif(1000)
> y <- runif(1000)
> sum(abs(x - y) < 1/2)/1000
[1] 0.743
```

Bonus: in a simulation of 1000 trials where the probability of a trial being less than 1/2 is $p = 3/4$ what is the expected number of trials to be successful? The variance?

Conditional probability 1. There are similar – but different – formulas for discrete and continuous

Exercise 8 *1. Suppose X and Y are uniform on 1,2,3,4. Find the p.d.f of $X+Y$ by computing*

$$f(k) = \sum P(X + j = k|Y = j)P(Y = j)$$

2. Suppose X and Y are uniform on $[0,1]$ – and independent

(a) Find $f(X = x|Y = y)$.

2a *We use the formula $f(x|y) = f(x,y)/f_Y(y)$ and independence to see:*

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x) = 1$$

(b) Use that to compute the density of $X+Y$

$$f_{X+Y}(x) = \int_0^1 f(X = x - y|y = y)f_Y(y)dy$$

2b *The integrand is 1 – but only when $x - y > 0$ as otherwise it is 0. (WHY?) This integral depends on x which runs from $0 to 2$. When $x < 1$ we have to worry. Here is what we get:*

If $x < 1$

$$\int_0^x 1 * 1 dy = x$$

if $1 < x < 2$ *then we can't have y be smaller than* $x - 1$ *as if it were, X couldn't be large enough to add to y to get x (it is not bigger than 1.*

$$\int_{x-1}^1 1 dy = 1 - (x - 1) = 2 - x$$

Independence Independence means multiply! For statistics it is a key example, as then the joint distribution of a sample from the same population will have joint density

$$f(x_1, x_2, ..., x_n) = f(x_1)f(x_2)\cdots f(x_n)$$

which with luck is manageable.

To check independence is different. We need to show for all cases that the joint p.d.f. is a product of the marginal.

Exercise 9 *1. For the following table:*

```
> m
    Y
X         -1      0      1
  1 0.0769 0.1538 0.0769
  2 0.1538 0.0769 0.1538
  3 0.0769 0.1538 0.0769
```

Are X and Y independent?

1 *No. If you know* $X = 1$ *then* -1 *and* 1 *are more likely for* Y *than* 0*, but otherwise if you know* $X = 0$ *it is reverse.*