1 Questions to be handed in on integration:

Read about this material here: integration.

When done with this project, you can print it by *first* selecting the parts to print, the printing the selection.

To get started, we load `Gadfly` so that we can make plots, and load the `Roots` package for `D`:

```
using Gadfly           # ignore any warnings
using Roots         # for D and fzero
using SymPy
```

1.0.1 Quick background Read the notes for more detail

In many cases, the task of evaluating a definite integral is made easy by the fundamental theorem of calculus which says that for a continuous function f that the following holds for any of its antiderivatives, F:

$$\int_a^b f(x)dx = F(b) - F(a).$$

That is the definite integral is found by evaluating a related function at the endpoints, a and b.

The `SymPy` package can compute many antiderivatives using a version of the Risch algorithm that works for *elementary functions*. For example. The area between 0 and 1 of the function x^2 is found with

```
x = Sym("x")
f(x) = x^2
integrate(f(x), x, 0, 1)
```

Of course, this could be done directly, as an antiderivative is just $x^3/3$:

```
F(x) = x^3/3
F(1) - F(0)
```

However, this only works *if* there is a known antiderivative $F(x)$. If not, what to do?

In this case, we can appeal to the definition of the definite integral. For continuous, non-negative $f(x)$, the definite integral is the area under the graph of f over the interval $[a, b]$. This area can be directly *approximated* using Riemann sums, or some other approximation scheme.

Imagine one didn't know, F. Then the Riemann approximation can help. The following pattern will compute the sum using right-hand endpoints:

```
f(x) = x^2
a, b, n = 0, 1, 5       # 5 partitions of [0,1] requested
delta = (b - a)/n       # size of partition
xs = a + (1:n) * delta
fxs = [f(x) for x in xs]
sum(fxs * delta)        # a new function ‘sum‘ to add up values in a container
```

That isn't very close to 1/3. But we only took $n = 5$. Bigger ns mean better approximations:

```
f(x) = x^2
a, b, n = 0, 1, 50_000      # 50,000 partitions of [0,1] requested
delta = (b - a)/n
xs = a + (1:n) * delta
fxs = [f(x) for x in xs]
sum(fxs * delta)
```

Note that only the first two lines need changing to adjust to a new problem. This makes it easy to wrap this in a function:

```
function riemann(f, a, b, n)
    delta = (b - a)/n
    xs = a + (1:n) * delta
    fxs = [f(x) for x in xs]
    sum(fxs * delta)
end
riemann(f, 0, 1, 50_000)
```

The Riemann sum is very slow to converge here. There are faster algorithms both mathematically and computationally. We will discuss two: the trapezoid rule, which replaces rectangles with trapezoids. Below we need both the left and right endpoints to compute the area of the trapezoid:

```
function trapezoid(f, a, b, n)
    delta = (b - a)/n
    x = a + (0:n) * delta
    trap(l,r) = (1/2)*(f(l) + f(r))
    ys = [trap(l,r) for (l,r) in zip(x[1:end-1], x[2:end])]
    sum(ys * delta)
end
trapezoid(f, 0, 1, 5000)
```

As well, we have Simpson's rule, which put a parabola at the top of the rectangle. Again, we need both left and right endpoints of each interval to compute:

```
function simpsons(f, a, b, n)
    delta = (b - a)/n
    x = a + (0:n) * delta
    simp(l,r) =  (f(l) + 4f((l+r)/2) + f(r))/6
    ys = [simp(l,r) for (l,r) in zip(x[1:end-1], x[2:end])]
    sum(ys * delta)
end
simpsons(f, 0, 1, 500)
```

Base `julia` provides the `quadgk` function which uses a different approach altogether. It is used quite easily:

```
f(x) = x^2
ans, err = quadgk(f, 0, 1)
```

This function returns two values, an answer and an estimated maximum possible error. The ans is the first number, clearly it is 1/3, and the estimated maximum error is the second. In this case it is small (10^{-17}) and is basically 0.

1.0.2 Questions

- Let $g(x) = x^4 + 10x^2 - 60x + 71$. Find the integral $\int_0^1 g(x)dx$ exactly using the fundamental theorem of calculus.

- For $f(x) = x/\sqrt{g(x)}$ (for $g(x)$ from the last problem) estimate the following using 1000 Riemann sums:

$$\int_0^1 f(x)dx$$

- Let $f(x) = \sin(\pi x^2)$. Estimate $\int_0^1 f(x)dx$ using 20 Riemann sums

- For the same $f(x)$, compare your estimate with 20 Riemann sums to that with 20,000 Riemann sums. How many digits after the decimal point do they agree?

Left Riemann The left Riemann sum uses left-hand endpoints, not right-hand ones. In the pattern above, this simply uses

```
xs = a + (0:n-1) * delta
```

instead of `xs = a + (1:n) * delta`.

- For $f(x) = e^x$ use the left Riemann sum with $n = 10,000$ to estimate $\int_0^1 f(x)dx$.

- The left and right Riemann sums for an increasing function are also lower and upper bounds for the answer. Find the difference between the left and right Riemann sum for $\int_0^1 e^x dx$ when $n = 10,000$. (Use your last answer.) What is the approximate value 1/100, 1/1000, 1/10000, or 1/100000?

trapezoid, Simpson's

- The answer to $\int_0^1 e^x dx$ is simply $e^1 - e^0 = e - 1$. Compare the error of the trapezoid method when $n = 10,000$.

- The answer to $\int_0^1 e^x dx$ is simply $e^1 - e^0 = e - 1$. Compare the error of the Simpson's method when $n = 10,000$.

(The error for Riemann sums is like $1/n$, the error for trapezoid sums is $1/n^2$ and that for Simpson's is $1/n^4$.)

1.1 quadgk

- Use `quadgk` to find $\int_{-3}^3 (1 + x^2)^{-1} dx$. What is the answer? What is the estimated maximum error?

1.1.1 Improper integrals

An *improper integral* is one which involves infinity one of few ways:

- one or both limits is infinite
- the function f has a vertical asymptote in the interval $[a, b]$ (e.g., $f(x) = 1/x$ on $[0, 1]$.

For these cases, the fundamental theorem of calculus does not apply, but the definite integral can be defined in terms of limiting values over sub-ranges. (For example, $\lim_M \int_0^M e^{-x} dx = \int_0^\infty e^{-x} dx$.)

Kahan (in a very interesting article about integration on a calcalculator) goes on to add these as troubling:

- the integrand oscillates infinitely rapidly in the interval $[a, b]$ (e.g., $f(x) = \sin(1/x)$ on $[-1, 1]$.
- the integrand or its first derivative changes wildly within a relatively narrow subinterval or oscillates frequently.

The `quadgk` function can handle these cases well, as we see through some examples

- The integral $\int_0^1 1/\sqrt{x} dx$ is an improper integral that is defined. What is the value?

- The integral $\int_0^1 x^{-2} dx$ is an improper integral that is not defined. How does `julia` report the error?

- The integral $\int_0^\infty e^{-x^2} dx$ is important in probability theory and many other areas. Compute its value with `quadgk`. (`Inf` is infinity.)

- The function $\sin(1/x)$ has no limit at 0, but the integral $\int_0^1 \sin(1/x)dx$ can be defined in terms of a limit. What is the value estimated by `quadgk`? Any thoughts on why this integral may take so long to be computed?

- Define

$$f(u) = \frac{\sqrt{u}}{u-1} - \frac{1}{\log u}$$

The improper integral $\int_0^1 f(u)du$ is defined. What is the value? What is the estimated error? Is this consistent with a value of 0.03649 plus or minus 0.00000007?

1.2 Applications

We discuss two applications of the integral . The first is a formula for a volume of revolution. For cylindrically symmetric volume, such as a drinking glass, if the radius as a function of height is given by $r(h)$, the the volume is $\int_a^b \pi r(h)^2 dh$.

Next is the length of the graph of $f(x)$ over the interval $[a, b]$. This is $\int_a^b \sqrt{1 + f'(x)^2} dx$.

For example, the arc-length of the sine curve over $[0, 2\pi]$ can be found with

```
using Roots          # for D
f(x) = sin(x)
g(x) = sqrt(1 + D(f)(x)^2)
quadgk(g, 0, 2pi)
```

1.2.1 arc length

- A stage manager is setting up chairs in a shape given by a parabola. She computes that the shape is given (in feet) by $g(x) = 10x^2$ for $-5 < x < 5$. If each chair takes approximately 2 feet, about how many chairs will be needed?

- The function $f(x) = x^2 - (1/8) \log(x)$ is called a pursuit curve, as it comes from solving how an efficient person would track someone running away from them on the y axis. What is the length of the curve between 1/10 and 10?

1.2.2 glass half full

- A glass is formed as a volume of revolution with radius as a function of height given the equation $r(h) = 2 + (h/20)^4$. The volume as a function of height b is given by $V(b) = \int_0^b \pi r(h)^2 dh$. Find $V(25)$.

- Find a value of b so that $V(b) = 455$.

- Now find a value of b for which $V(b) = 455/2$. This height will have half the volume as the height just found. Compare the two values. Is the ratio of smallest to largest 1/2, more than 1/2 or less?