1 Questions to be handed in for project 6:

Read about this material here: Approximate derivatives in julia.

When done with this project, you can print it by *first* selecting the parts to print, the printing the selection.

To get started, we load `Gadfly` so that we can make plots.

```
using Gadfly            # ignore any warnings
```

1.0.1 Quick background Read the notes for more detail

The slope of the tangent line to the graph of $f(x)$ at the point $(c, f(c))$ is given by the following limit:

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

The notation for this – when the limit exists – is $f'(c)$, the intuition is that this is the limit of the slope of a sequence of secant lines connecting the points $(c, f(c))$ and $(c+h, f(c+h))$. In general the derivative of a function $f(x)$ is the function $f'(x)$, which returns the slope of the tangent line for each x where it is defined.

Approximating the tangent line can be done several ways. The *forward difference quotient* takes a small value of h and uses the values $(f(x+h) - f(x))/h$ as an approximation. In `julia` we can write a function that does this:

```
forward(f, x; h=1e-6) = (f(x+h) - f(x))/h
```

We can define an *operator* – something which takes a function and returns a function like this:

```
Df(f; h=1e-6) = x -> (f(x+h) - f(x))/h
```

The right-hand side returns an anonymous function (the arrow). This can be used to approximate the derivative of $f(x)$. For convenience, we define the prime notation – so that we can just use `f'(x)` – with this trick:

```
Base.ctranspose(f::Function) = Df(f)
f(x) = sin(x)
f'(pi)              # compare to cos(pi)
```

In the `Roots` package, an operator `D` is given which uses a numeric approach to compute the derivative. This is more accurate, but conceptually a bit more difficult to understand and does not work for all functions. It is used like an operator, e.g., `D(f)` is a function derived from the function `f`:

```
using Roots
fp(x) = D(sin)(x)       # define a function fp or use D(sin) directly
fp(pi)                  # finds cos(pi)
```

1.0.2 Questions

- Calculate the slope of the secant line of $f(x) = 3x^2 + 5$ between $(2, f(2))$ and $(5, f(5))$.

- Verify that the derivative of $f(x) = \sin(x)$ at $\pi/3$ is $1/2$ by finding the following limit using a table:

$$\lim_{h \to 0} \frac{f(\pi/3 + h) - f(\pi/3)}{h}$$

(Use `[hs ys]` to look at your generated data, as was done in the limits assigment.)

- Let $f(x) = 1/x$ and $c = 2$. Find the approximate derivative when `h=1e-6`.

- Let $f(x) = x^x$ and $c = 3$. Find the approximate derivative when `h=1e-6`.

- Let $f(x) = (x+2)/(1+x^3)$. Plot both f and its approximate derivative on the interval $[0, 5]$. Identify the zero of the derivative. What is its value? What is the value of $f(x)$ at this point?

- Let $f(x) = (x^3 + 5)(x^3 + x + 1)$. The derivative of this function has one real zero. Find it. (You can use `fzero` with the derivative function after plotting to identify a bracketing interval.)

- Let $f(x) = \sin(x)$. Following the example on p124 of the Rogawski book we look at a table of values of the forward difference equation at $x = \pi/6$ for various values of h. The true derivative is $\cos(\pi/6) = \sqrt{3}/2$.

  Make the following table. What size `h` has the closest approximation?

```
f(x) = sin(x)
c = pi/6
hs = [(1/10)^i for i in 1:12]
ys = [forward(f, c, h=h) for h in hs] - sqrt(3)/2
[hs ys]
```

- The D operator is easy to use. Here is how we can plot both the sine function and its derivative

```
using Gadfly, Roots         # to load plot and D
f(x) = sin(x)
plot([f, D(f)], 0, 2pi)
```

Make a plot of $f(x) = \log(x+1) - x + x^2/2$ and its derivative over the interval $[-3/4, 4]$. Is the derivative always increasing?

- The function $f(x) = x^x$ has a derivative for $x > 0$. Use `fzero` to find a zero of its derivative.

- Higher-order derivates can be approximated as well. One can use, say, `D(f,2)` of `f''(x)` to approximate the second derivative. (The latter uses the forward difference approximation twice.) Look carefully at the output of these commands. Do they show that the two give similar results?

```
f(x) = sin(x)
g(x) = D(f,2)(x) - f''(x)   # difference of different approximate derivatives
map(g, linspace(0, pi, 10))
```

Now replace the line `g(x) = D(f,2)(x) - f''(x)` with a comparison of 4th derivatives `g(x) = D(f,4)(x) - f''''(x)`. Do you get the same level of approximation? In other words, would you want to use `f''''(x)` to find a fourth derivative? (Assume `D(f,4)` is the correct answer.)

1.1 some applications

- Suppose the height of a ball falls according to the formula $h(t) = 300 - 16t^2$. Find the rate of change of height at the instant the ball hits the ground.

The tangent line to the graph of $f(x)$ at $x = c$ is given by $y = f(c) + f'(c)(x - c)$. It is fairly easy to plot both the function and its tangent line. This function is a helper, though it can be done directly:

```
f(x) = x^2          # replace me
tangent(f, c) = x -> f(c) + D(f)(c)*(x-c) # returns a function
plot([f, tangent(f, 1)], 0, 2)
```

- For the function $f(x) = 1/(x^2 + 1)$ (The witch of Agnesi), graph f over the interval $[-3, 3]$ and the tangent line to f at $x = 1$.

- Let $f(x) = x^3 - 2x - 5$. Find the intersection of the tangent line at $x = 3$ with the x-axis.

- Let $f(x)$ be given by the expression below. Verify that the

```
f(x; a=1) = a * log((a + sqrt(a^2 - x^2))/x ) - sqrt(a^2 - x^2)
```

The value of `a` is a parameter, for which $a = 1$ is fine.
For $x = 0.25$ and $x = 0.75$ the tangent lines can be drawn with

```
plot([f, tangent(f, 0.25), tangent(f, 0.75)], 0, 0.8)
```

Verify that the length of the tangent line between $(c, f(c))$ and the y axis is the same for $c = .25$ and $c = 0.75$. (You can do this by making an appropriate right triangle whose ratio opp/adj is related to the derivative, the length is the hypotenuse of this triangle.)

- A formula for blood alcohol level in the body based on time is based on the number of drinks and the time wikipedia.

Suppose a model for the number of drinks consumed per hour is

```
n(t) = t <= 3 ? 2 * sqrt(3) * sqrt(t) : 6.0
```

Then the BAL for a 175 pound male is given by

```
bal(t) = (0.806 * 1.2 * n(t)) / (0.58 * 175 / 2.2) - 0.017*t
```

From the plot below, describe when the peak blood alcohol level occurs and is the person ever in danger of being above 0.10?

```
plot([bal, bal'], .5,6)
```