[Disclaimer: this study sheet is not intended to contain all the possible questions that may appear on the test. It is only intended to help you get started on studying for the exam.]

[Disclaimer 2: I gave a range of problems to look at, not expecting you to try all of them. Do try some (many are in the HW).]

The third test in MTH 231 will cover the following lessons from the syllabus: 28-42, or sections 3.3 - 4.4. The material in chapter 4 will only be chosen from what we covered in class as of 11/26. (This should allow you to compute a definite integral using the fundamental theorem of calculus.)

The test will be for a full period and will have 100 possible points in roughly 20 questions.

What to study? Here is an overview of the material with a few sample questions indicated from the book, or written by me.

First derivative test This was covered at the end of the last test, but is so essential I'm going to have a question or two on it again. The basic idea is covered in this table:

```
f(x)       |    f'(x)
-----------------------
increasing |  positive
decreasing |  negative
```

The actual facts are a tad more subtle (f can increase on an interval, but f' can be 0 at a few points along the way.)

You need to know how to use this fact to characterize whether a critical point is a relative max or relative min.

Two skills are needed here: taking a derivative and making a sign table.

Some sample questions: 3.3 24, 36, 29, 55, 74, 95-105

Concavity, 2nd derivative test You need to understand visually what we mean by concave up and convave down. As well, what an inflection point looks like.

The basis of concavity for functions with a second derivative is summarized by this table:

```
   f(x)           f'(x)         f''(x)
---------------------------------------
concave up      increasing     positive
concave down    descreasing    negative
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You should have a clear idea why this is so (the second part is just the table above applied to the function $f'(x)$.)

The second derivative test takes advantage of the fact that a relative min occurs when a function is concave up, and a relative max occurs when a function is concave down. It provides an alternative to the first derivative test when applicable.

The main skills here are taking a second derivative and again the sign table.

Some sample questions: any in exercises 11-26 of section 3.4; ditto for 27-40; some thinking ones on 45-47, 58; 79-82

Limits at infinity These complement limits of infinity (as presented in class.) The basic idea is to think about a function for really large values of x (or really large negative values of x). The key facts that guide you are

- $1/x^n$ will go to 0 for really large x $(n > 0)$
- x^n will go to infininty faster than x^m if $n > m$.

So for polynomials the leading term (highest power) tells you what the large x behaviour is. (So the graph of $-x^2 + ax + b$ will look like the graph of $-x^2$ for large x). The ratios of polynomials can be considered by comparing the leading terms of each.

Another useful thing to keep in mind is that problems involving sin or cos these terms can usually be replaced by a 1 when you think about them, as that is the largest these values can be. For instance, $\sin(x)/x$ has a limit at **infinity** of 0 (**Not** 1 which is the limit at 0!)

There are no real touch skills here – just the task of identifying the leading term.

Some sample problems from 3.5: any of 17-35; 51 is a good one to think through as is 87 and 88.

Curve sketching Curve sketching is a way of combining many things you've learned in math:

1. Skills from pre-calculus (domain, range, symmetry, x-intercepts, y-intercepts)
2. Skills from calculus:
   (a) increasing/decreasing as indicated by the first derivative
   (b) relative max, relative min as given by first derivative test (or second)
   (c) Concavity of the function, as indicated by the second derivative's signe
   (d) inflection points – or when the concavity changes sign
   (e) Horizontal and vertial asymptotes – these are from the limits at infinity and limits of infinity respectively.

Recall the value is not just producing a graph (a computer can do that) but producing a sketch that focuses on the main/important features of the graph.

The main skill is to generate the sign table for both $f'(x)$ and $f''(x)$. But you must also find limits at infinity, solve for $f(x) = 0$, and *think hard* to put everything together.

Some sample problems to look at from the section: any of 7-34, 67-70, 47-50

Optimization Problems The use of calculus to optimize a function (find the minimum or maximum over some interval) is one of the more practically useful things taught. Unfortunately the problems are hard to do. The main steps to an optimization problem are many:

1. Identify equations that relate unknown variables. Usually in these problems there are two: one giving a constraint between the variables that allows you to express one value in terms of the other; and a second that is the function you wish to optimize.
2. Figure out from the equations, a function of single variable that you wish to optimize over an interval (the interval must usually be thought through too.)
3. To optimize over a closed interval, we know the value will happen either at an end point or a critical point. As such we need to solve $f'(x) = 0$ or $f'(x)$ DNE. to find the critical points
4. Then, you decide the largest value either by evaluating the function at all the critical points and endpoints – or thinking hard to rule things out.
5. Finally, you answer the question as asked, which may involve the maximum value or when the maximum value occurs etc.

So there are two keys: setting up the problem; solving the problem completely

The main skill are setting up the problems. The calculus skills are important but covered in sections 3.1 and 3.3.

Let's look at 3-8. Here are the setup and the functions to optimize for the odd ones:

3 sum is 5: $a + b = 5$; product is maximum: $f = ab$. Solve for a to get this as $f(b) = (5 - b)b$. The interval would be $[0, 5]$. If b were bigger, then a would be negative. (You can quibble that positive means $(0, 5)$.)

5 Product is 192: $ab = 192$, sum of first plus three times the second is a minimum: $f = a + 3b$, solve for a to get $f(b) = 192/b + 3b$, Here b can be anything between 0 and infinity.

7 sum of first and second is 100: $a + b = 100$, product is a max: $f = ab$, which turns into $f(b) = (100 - b)b$. The interval is $[0, 100]$, as otherwise a would be need to be negative due to the constraint.

The problem 23, 26, 28-30 are all straight from the "greatest hits of calculus" playbook. Heres how you set up 23 and 26 – you can do the calculus part:

For 23 you have a constraint due to the perimeter: $x+2y+(2\pi x/2)/2 = x+2y+\pi x/2 = 16$. The function to maximize is $f(x) = xy + \pi(x/2)^2$. We should solve for y in the constraint, as x appears twice in the function. We have $y = (16 - x - \pi/2x)/2$ so $f(x) = x(16 - x - \pi/2x)/2 + \pi(x/2)^2$. This is quadratic function in x so better have a negative leading coefficient if it to have a maximum not occuring at an endpoint. As for these endpoints, the value of x must be 0 or more and no larger than what would happen when $y = 0$ (The window would be a half circle so $\pi(x/2)^2 = 16$ or $x = 8/\sqrt{pi}$. (See why this is popular – its a challenge)

For 26, we have through a) the area is 1/2 base times height. The height is $4 + h$, the base is twice the legnth of the missing isde of the right triangle with hypotenuse 4 and side h. So is $\sqrt{16 - h^2}$. Altogher, you are maximing $f(h) = 1/2(4 + h)2\sqrt{16 - h^2}$ on the interval $[0, 4]$.

Newton's method The only thing here I would test on is finding an equation of the tangent line to a graph of $f(x)$ at $x = c$ and then finding this intersection point. In fact – I already tested you on this last time.

That's not to say this isn't a real important problem. Your cell phone likely wouldn't work without the ability to solve $f(x) = 0$ numerically and this is the basis for numerical solutions to the equation. It is just hard to test on without a computer.

Differentials As mentioned in class, this is a real important concept – replacing a complicated function by a simpler function (linear one) that *approximates* the function well near some value $x = c$. Newton's method takes advantage of this, engineers do repeatedly. Again, your cell phone likely wouldn't work if such approximations weren't well known. (Antenna calculuations are hard to do exactly, but can be handled by approximations...)

As a bonus, in this section you learn **more notation** – yeah. The differential notation appears in integration and differential equations.

Some problems to do: any of 11-20 to find differential; any of 27 to 32 to use. (These problems involve error in measurement); any of 43-46.

Integration I presented integration differently than the book: I spoke about area first (4.2); then Riemann Sums and Definite Integrals (4.3); then the first Fundamental Theorem of Calculus (4.4). This motivates the need for antiderivatives and the indefinite integral (4.1). The test will cover all of this (half way through 4.4, with the key formula

$$\int_a^b f(x)dx = F(b) - F(a).$$

The key concept is that we can find approximations to the area under a curve by using rectangles. This approximation gets better and better as the base of the rectangles goes to 0. In the limit the approximation gives the area, which we denote by the definite integral:

$$\lim \sum f(c_i)\delta x_i = \int_a^b f(x)dx.$$

To compute that area we can use the sum (but that is only practical numerically with a computer) **or** we can learn some facts about integration that help us find the value. These facts are:

1. The FTC:
$$\int_a^b f(x)dx = F(b) - F(a).$$
2. The rule for constants:
$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$
3. The rule for sums
$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$
4. Some rules based on area:
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$
5. And the reverse power rule.

Some problems to consider 4.1 15-42. (I'm skipping the discussion of differential equations until after the exam.)

Section 4.2 1-6; 7-12; 15-20 (practice on summation notation) 23-26; 31-33; 47, 48

SEction 4.3: 9-12; 13-18; 23-32 (the latter are "set-up" problems. Thought problems 63-69

Section 4.4: 5-26; 27-32; 33-38; 39-42.