Modern algebra is about abstracting out the key features of different algebraic systems. This worksheet is more or less a review of some different mathematical structures you have met along your way.

Arithmetic

Consider the set of integers $Z = \{\ldots,-2,-1,0,1,2,\ldots\}$ with addition.

1. Is there a number y (in Z!) such that $x+y=x$ for all possible x? If so, what is it? Call this e.
2. For a given x, is there **always** a number y such that $x+y=e$?
3. Is it true that $x+y+z=(x+y)+z=x+(y+z)$? What is this property called. (What does it mean "is it true" mathematically speaking?)

Consider the set of integers $Z = \{\ldots,-2,-1,0,1,2,\ldots\}$ with subtraction.

1. Is there a number y such that $x-y=x$ for all possible x? If so, what is it? Call this e.
2. For a given x, is there **always** a number y such that $x-y=e$?
3. Is it true that $x-y-z=(x-y)-z=x-(y-z)$? Why is this different than the previous example.

Consider the clock numbers $Z_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$ with addition (addition is modulo 12).

1. Is there a number y such that $x+y=x$ for all possible x? If so, what is it? Call this e.
2. For a given x, is there **always** a number y such that $x+y=e$? What is y when x is 8?
3. Let $<x>$ denote the set $\{x,2x,3x,\ldots\}$. What is the size of $<3>$ and $<5>$. Why are they different?

We know that every positive integer can be written in a unique way as a product of prime factors. For instance $12 = 2^2 \cdot 3$. We say that m is relatively prime to n if they have no prime factors in common.

Let U be the set $\{1,3,7,9\}$. These are the integers between 1 and 10 which are relatively prime to 10. Let $\times$ be *multiplication modulo 10*. That is $a \times b = (ab) mod(10)$. Fill in the following multiplication table

- Is there a number e so that for each x $x \times e = x$?
- For each x is there a number y with $x \times y$ = e?
- Is it always so that $(x \times y) \times z = x \times (y \times z)$?
- What are the sizes of $<1>$, $<3>$, $<7>$, $<9>$?

$\times$	1	3	7	9
1				
3				
7				
9				

Table 1: multiplication table

Let V be the set $\{0,1,2,3\}$ and $\times$ be multiplication modulo 4.

- Is there a number e so that for each x $x \times e = x$?
- For each x is there a number y with $x \times y =$ e? If not, for which x is this not so?

Let Z be the integers under multiplication, $\cdot$.

- Is there a number e so that for each x $x \cdot e = x$?
- For each x is there a number y with $x \cdot y =$ e? If not, for which x is this not so?

Consider the set of polynomials $= \{a + bx + cx^2\}$ for a, b, c integers under addition of polynomials.

- Is there a polynomial e so that for each x, $x + e = x$?
- For each x is there a number y with $x + y =$ e? If not, for which x is this not so?
- Is it true that if x and y are in the set then $x + y$ is in the set?

Consider the set of polynomials $= \{a + bx + cx^2\}$ for a, b, c integers under *multiplication* of polynomials.

- Is there a polynomial e so that for each x, $x + e = x$?
- For each x is there a number y with $x + y =$ e? If not, for which x is this not so?
- Is it true that if x and y are in the set then $x + y$ is in the set?

Consider the set of all polynomials $p(x) = a_0 + a_1 x + \ldots a_n x^n$ of arbitrary degree with integer coefficients. Now suppose that $x^2 = -1$. Find a polynomial of the type $a + bx$ that is equivalent to $1 + x + x^2 + x^3 + x^4$. Are $1 - x + x^2 - x^3$ and $-1 + x - x^2 + x^3$ equivalent?

Let M be the set of 2 by 2 matrices with real coefficients, and I be the identity matrix.

- Is there a value e in M so that $x + e = x$ for any x in M?
- Is there a value e in M so that $x \times e = x$ for any x in M?
- For each x in M is there a y in M so that $x + y = 0$?

- For each x in M is there a y in M so that $x \times y = I$?
- We know that matrix multiplication is associative: $(AB)C = A(BC)$ What about commutative?
  - If you can, find two matrices A and B such that $AB = BA$?
  - If you can, find two matrices A and B such that $AB \neq BA$?

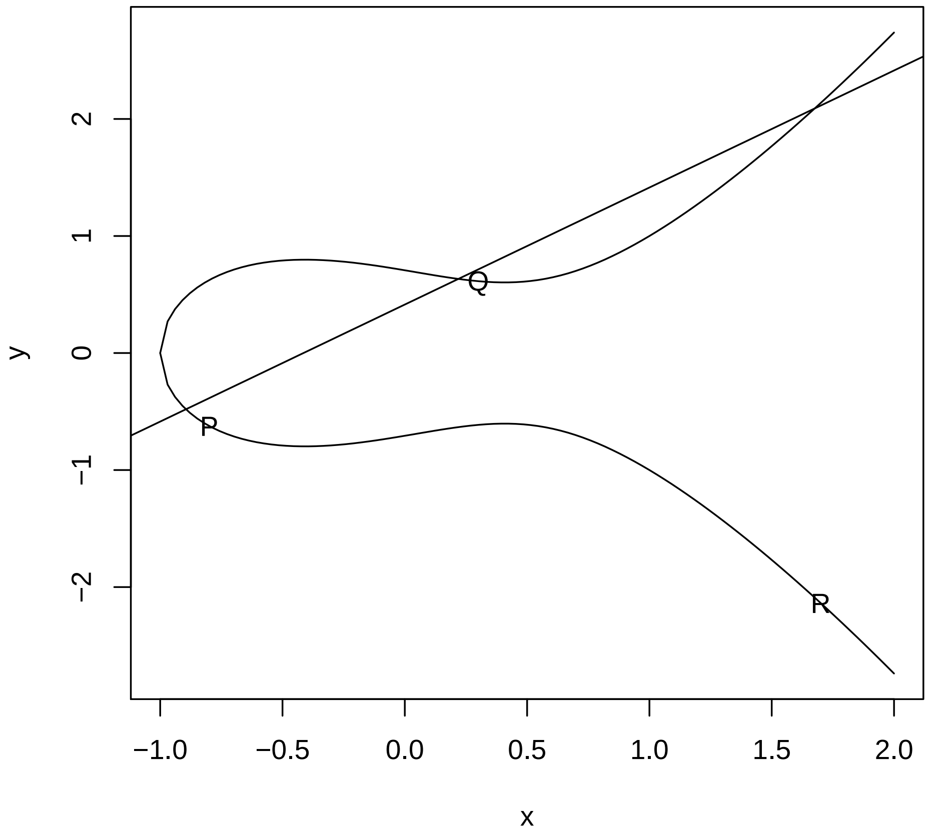
Let A be the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- How big is $< A >= \{A, A^2, A^3, \dots\}$ under matrix multiplication?
- One can view A as a rotation, as Ax is the vector x rotated 90 degrees counter clockwise. Would this have helped you find $< A >$ more quickly? How?
- For each matrix, x, in $< A >$, there is a matrix y with $xy = I$. Find all of these.

Let A be 2 by 2 matrices with integer entries under multiplication. Find two non-zero matrices with $AB = 0$. What is the determinant $(ad - bc)$ of your matrices?

(Wierd example). Consider the points on the curve $y^2 = x^3 - x/2 + 1/2$ and define addition as follows. For a point $P = (a, b)$ define $-P = (a, -b)$. Then if P, Q, R all lie on a straight line, then $P + Q = -R$. See the figure



- The “point at infinity” is a special point. Explain how $R+(-R)$ is defined by this point at infinity?
- Is this addition associative?
- Is it true that $P+Q = Q+P$?
- Is $P+Q$ always defined, if one uses the “point at infinity?”