1) in $\mathbb{Z}_{16}$ there are $\phi(2)=1$ elements of order 2 (8)

in $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ look at $|(a,b)| = \mathrm{lcm}(|a|,|b|) = $

if $=2$ then $|a|=1$ $|b|=2$

or $|a|=2$ $|b|=1$

or $|a|=2$ $|b|=2$

in either case only one solution $(0,1),(4,0),(4,1)$

so 3 elements

in $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ same argument $\left((2,0),(0,2),(2,2)\right)$

2) $|G|=45=3\times 15$

if Abelian $G \approx \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$

or $\mathbb{Z}_9 \oplus \mathbb{Z}_5$

in first case $H=\langle(0,1,1)\rangle \approx \mathbb{Z}_{15} \subseteq G$

in second case $H=\langle(3,1)\rangle \approx \langle 3\rangle \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{15} \subseteq G$.

3. $G=U(91)$ H the group $|H|=9$, H Abelian so

$H \approx \mathbb{Z}_9$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3$

Since $G \approx U(7) \oplus U(13) \approx \mathbb{Z}_6 \oplus \mathbb{Z}_{12}$ has no elements of order 9

$H \approx \mathbb{Z}_3 \oplus \mathbb{Z}_3$

5 ~~Zan~~

if $G \approx \mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_2^{n}} \cdots \oplus \mathbb{Z}_{p_k^{n_k}}$

Then $G \approx \mathbb{Z}_{p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}}$ <u>if</u> $p_1, p_2 \ldots p_k$ are relatively prime.

so need the $p_i$ to be distinct.

6. $G$ Abelian, $|G|$ odd

~~Say $G \approx \mathbb{Z}_{2}$~~ Say $G \approx \mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_2^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}}$

if $a \in G$ has $|a| = 2p$ then

$a = (a_1, a_2, \ldots a_k)$

has $2p = \text{lcm}(|a_1|, \ldots, |a_k|)$ so $2 \mid |a_i|$ for some $i$

so $2 \mid |\mathbb{Z}_{p_i^{n_i}}|$ which is true only if $p_i = 2$

or $G$ is even. $\times$

- $\phi(r^i f^j) = j \bmod 2$.
- $\phi$ is onto $\phi(r) = 0$ $\phi(f) = 1$.
- $\phi(r^i f^j r^\ell f^m) = \phi(r^i r^{-\ell} f^{j+m}) = j+m \bmod 2$

$\phi(r^i f^j)\phi(r^\ell f^m) = j + m$ in $\mathbb{Z}_2$ $= j+m \bmod 2$

kernel $\phi = \{ r^i f^j : \phi(r^i f^j) = 0 \}$

$= \{ r^i \} = \langle r \rangle$.

2. show ~~closure~~ if $g h \in \phi(K)$ then $g h^{-1} \in \phi(K)$

Now there are $g', h'$ w/ $\phi(g') = g$ $\phi(h') = h$, $g', h'$ in $K$

So $\cancel{\phi(g}$ $gh^{-1} = \phi(g')\phi(h')^{-1}$

$= \phi(g')\phi(h'^{-1})$

$= \phi(g' h'^{-1})$

$= \phi(k)$ some $k$ in $K$

in $\phi(K)$

3. $\mathbb{Z}_n \cong \mathbb{Z}/\langle n \rangle = \{ 0 + \langle n \rangle, 1 + \langle n \rangle, \ldots, n-1 + \langle n \rangle \}$

as $(a + \langle n \rangle) + (b + \langle n \rangle) = a + b + \langle n \rangle$

$= a + b \bmod n + \langle n \rangle$

Take $\phi : \mathbb{Z}/\langle n \rangle \to \mathbb{Z}_n$ by $a + \langle n \rangle \to a \bmod n$.

4. use $\phi(j) = j \bmod k$.

1. $H \le G$, $G:H = 2$ show $H \trianglelefteq G$.

~~$aHa^{-1} = H$ or $aH$~~

~~Take G~~

we have $G = H \cup aH$ (2 cosets).

let $x$ be in $G$ if $x$ in $H$ then $xHx^{-1} = H$.

If $x$ in $aH$ then $x = ah$. for some $h$

take $h'$ in $H$

$$ah(h')(ah)^{-1} = ahh'(h^{-1})a^{-1}$$
$$= a\tilde{h}a^{-1}$$

if this is in $H$ we're good as $xHx^{-1} \subseteq H$.

Suppose not. Then $a\tilde{h}a^{-1} = ah'$

or $\tilde{h}a^{-1} = h^{-1}$

or $\tilde{h}h = a$ so $a$ in $H$ ~~X~~ as $aH \neq H$.

2. $14 + \langle 8 \rangle$ in $\mathbb{Z}_{24}/\langle 8 \rangle$

we have $14 + \langle 8 \rangle = 6 + \langle 8 \rangle$

$(6 + \langle 8 \rangle) + (6 + \langle 8 \rangle) = 12 + \langle 8 \rangle = 4 + \langle 8 \rangle$

$(6 + \langle 8 \rangle) + (4 + \langle 8 \rangle) = 10 + \langle 8 \rangle = 2 + \langle 8 \rangle$

$(6 + \langle 8 \rangle) + (2 + \langle 8 \rangle) = 8 + \langle 8 \rangle = 0 + \langle 8 \rangle$

So $|14 + \langle 8 \rangle| = 4$

3. $H \times K$ is Abelian; as

1) $H \times K \cong H \oplus K$
2) $H \oplus K$ is abelian if $H, K$ are.

4. Same reason, as any non-trivial subgroup of $D_4$ is Abelian as it has order 2 or 4.

5. Using the fact $G/_{Z(G)} \cong \text{Inn}(G)$

if $Z(G) = e$ (since there are no normal subgroups and $G$ not Abelian)

we have $A_n \cong \text{Inn}(A_n)$ if $\underline{\underline{n \geq 5}}$.

6. $H = \{h : h(i) = i$ if $i$ is a corner piece$\}$

Fact $G$ maps corner pieces to corner pieces.

So if $g \in G$, $i$ a cornerpiece then

$$ghg^{-1}(i) = gh(j) \quad \text{where} \quad j = g^{-1}(i)$$
$$\text{or} \quad g(j) = i.$$

Now $j$ is also a corner piece so $h(j) = j$ so $g \circ h \circ g^{-1}(i) = g(j) = i$. That is $ghg^{-1}$ is in $H$

So $gHg^{-1} \subseteq H$ and $H$ is normal.