Counting

For counting problems we have 3 basic formulas

- The counting principle which states if there are n_i choices at stage i, then the total number of choices for all stages is

$$n_1 \cdot n_2 \cdots n_j.$$

- The number of *permutations* of size r from n *distinct* objects is

$${}_nP_r = n \cdot (n-1) \cdot \cdots \cdot (n-r+1) = \frac{n!}{(n-r)!}.$$

  A permutation is a selection where the order of selection is important.

  A special case is a permutation of size n or simply a permutation or reordering. There are $n!$ of these.

- The number of *combinations* of size r from n distinct objects is "n choose r" or

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}$$

  A combination is a selection where the order is not important. For example, when dealing cards or the lottery numbers.

Sometimes these need to be combined to answer a problem.

To implement these on the computer, you need to learn two functions: `prod()` to multiply terms together and `choose()` to implement the formula for combinations.

1 Using `prod()` to find products

A new car has many options. Suppose you have 8 paint colors, 3 interior colors, 2 interior fabrics, and 2 choices for transmission. How many different cars are possible?

The answer is found by multiplying all the choices at each stage. So it is

```
> 8 * 3 * 2 * 2
```

```
[1] 96
```

A function to multiply all the numbers in a data vector is `prod()`. The above could have been done by storing the values and then multiplying

```
> x = c(8, 3, 2, 2)
> prod(x)
```

```
[1] 96
```

2 Using prod() to find factorials

There is no factorial function built in, but we can use the `prod` function to find factorials.

For example, to find 3! we can enter in the numbers 1, 2, 3 and then multiply with `prod()`

```
> x = c(1, 2, 3)
> prod(x)

[1] 6
```

This is a little silly as we entered the numbers 1, 2, 3 by hand. Its better to let the computer find sequences for us. Expressions like `a:b` will give a range of numbers from `a` to `b` separated by 1.

```
> 1:3

[1] 1 2 3

> 1:10

 [1]  1  2  3  4  5  6  7  8  9 10

> 10:1

 [1] 10  9  8  7  6  5  4  3  2  1

> 10:(10 - 5 + 1)

[1] 10  9  8  7  6
```

So to find a factorial is easy. We can find 7! by

```
> prod(1:7)

[1] 5040
```

An for any n, we can assign it first, then find the factorial

```
> n = 10
> prod(1:n)

[1] 3628800
```

Question 1: Find the following factorials

1. Find 69!
2. Find 70!. Why do you think 69! is the largest you can find on most calculators?

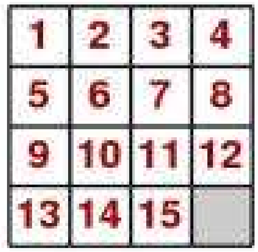

Figure 1: The game of 15

For a concrete example, the game of fifteen (Figure 1) can have many different rearrangements. At first guess there would be 16 choices for the upper corner, 15 for the next corner, etc. Altogether $16!$

Question 2: In the game of 15 we saw that the number of rearrangements appears at first glance to be $16!$. How big is this? Is it millions? trillions? billions? (We say "first glance" as there are really only $1/2$ these possibilities due to parity.)

Question 3: In a class of 30 students, the number of different arrangements for the top 5 students is ${}_{30}P_5$. What is this number? (Divide factorials)

3 Using `choose()` for binomial coefficients

The `choose()` function will compute $\binom{n}{k}$. For example, there are "52 choose 5" different poker hands when counted without order. This number is found with

```
> choose(52, 5)

[1] 2598960
```

That's over 2.5 million.

Question 4: In a lottery with 54 balls 6 are chosen. How many different combinations are possible?

Question 5: A group of 10 basketball players splits into two teams of 5. How many different ways can this be done?

Question 6: There are 8 places to sit down and 3 students. How many different combinations of seats can be chosen?

Question 7: Figure 2 randomly colors 15 of the 25 balls black, and the remaining 10 gray. How many different figures could have be produced?

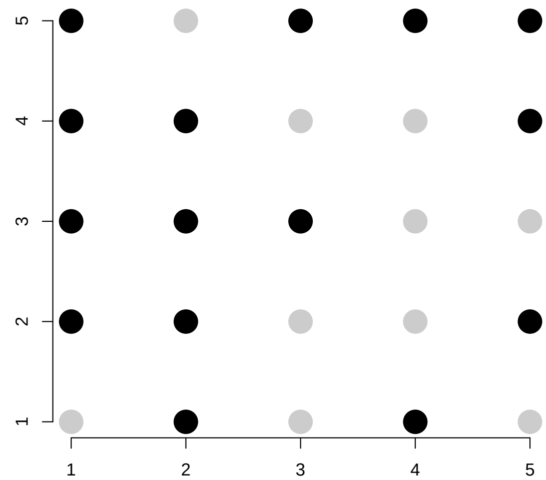

Figure 2: 10 balls colored black at random

4 Combining formulas

Sometimes the formulas needs to be combined to answer a question. For example, how many poker hands are a full house. A full house has the pattern *aaabb*, or a pair and three of a kind.

To count these, we use two stages. The first, is to select the three of a kind. The second to select the pair. We then combine the numbers using the product rule.

For the first stage, there are 13 ways to choose the face of the card (the a in the pattern). For each face there are $\binom{4}{3}$ ways to pick the suits. So all total there are $13 \cdot \binom{4}{3}$ ways.

For the second stage, there are 12 ways to choose the face of the card. (Why not 13?) Then for the given face, $\binom{4}{2}$ ways to pick the faces. Altogether there are

$$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}$$

ways.

Question 8: How many ways are there to have a full house? What is the probability of being dealt 5 cards and having a full house?

Question 9: Three of a kind is the pattern *aaabc*. Explain why the number of ways to have this pattern is

$$13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot \binom{4}{1} \cdot \binom{4}{1}.$$

(That is, why is it $13 \cdot \binom{12}{2}$ and not $13 \cdot 12 \cdot 11$?)

How many ways is this? How many more ways are there to have a three of a kind than a full house?

Question 10: A basketball team has 3 centers, 4 forwards and 5 guards. A starting lineup has one center, 2 forwards and 2 guards. How many starting lineups are possible?

5 Lotteries

Is a state lottery deserving of the nickname "stupid tax"? Let's compute some probabilities to see.

For example, a lottery has 54 balls, of which you mark off 6 as yours. The state chooses 6 balls. How many ways are there to match exactly 2 balls?

We count in two stages, first how many ways to match 2 of our 6 marked ball, then how many ways to match 4 of the unchosen balls. By the counting formula we multiply together to get the answer. Each one is seen as a combination problem. So there are $\binom{6}{2}$ ways to pick the 2 that match, and $\binom{48}{4}$ ways to pick the ones that don't match. (These numbers come from $48 = 54 - 6$, $4 = 6 - 2$.)

So there are $\binom{6}{2} \cdot \binom{48}{4}$ ways to match exactly 2 balls and the probability of matching exactly 2 is

$$\frac{\binom{6}{2} \cdot \binom{48}{4}}{\binom{54}{6}}$$

Question 11: What are these two numbers, the number of ways to match exactly 2 balls and the probability of the same.

Question 12: Repeat the above with matching exactly 5 balls.

The formula to match exactly k balls can be implemented as in this example

```
> k = 2
> choose(6, k) * choose(54 - 6, 6 - k)
[1] 2918700
```

Question 13: For $k = 0, 1, 2, 3, 4, 5, 6$ find the number of ways to match k balls. Which is most common? Do the above commands, except replace `k=2` with

```
> k = 0:6
> names(k) = k
```

Question 14: The game of pick 10 has 80 balls of which you pick 10, then 20 balls are selected. Why is the number of picks that match k balls given by

$$\binom{10}{k} \cdot \binom{80-10}{20-k}?$$

Question 15: Using `k=0:10` find the most common number k for the number of balls chosen. (The `max()` function will find the largest value.)