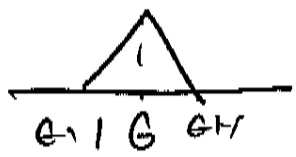
Answers to selected HW ...

10-3 $f(x|\theta) = 1 - |x-\theta|$

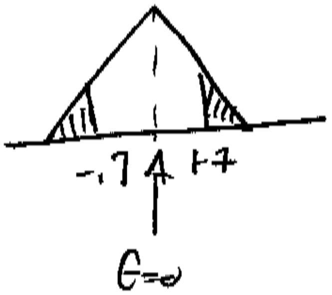


observed $X = .7$

$H_0: \theta = 0 \quad H_A: \theta \neq 0.$

so p-val $= P[X$ more extreme than observed $| H_0]$

$= P[X > .7$ or $X < -.7 \mid \theta = 0]$

$=$ 

$= 2 \times \underbrace{.3 \times .3}_{\text{area of triangle}}$

10-5

pop ~ Ber(p) $\quad f(x|p) = p^{x}(1-p)^{1-x}$

$L(p) = \prod_{i=1}^{n} f(x_i|p) = p^{\sum x_i}(1-p)^{\sum(1-x_i)} = p^{n\bar{x}}(1-p)^{n(1-\bar{x})}$

where $\bar{x} = \hat{p}$ (same thing).

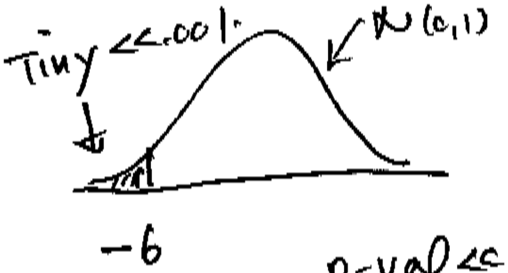
a) $\dfrac{L(.5)}{L(.9)} = \dfrac{(.5)^{n\bar{x}}(1-.5)^{n(1-\bar{x})}}{(.9)^{n\bar{x}}(1-.9)^{n(1-\bar{x})}} = \left(\dfrac{.5}{.9}\right)^{n\bar{x}}\left(\dfrac{.5}{.1}\right)^{n(1-\bar{x})}$

b) extreme if $H_0: p = .5 \quad H_A: p = .9$? larger values are extreme. The likelihood $L(p)$ is maximized at $\hat{p}$, so we expect $\hat{p}$ to be close to p and if $p = .9$ we'd expect $L(.9) > L(.5)$.

10.8 $H_0: \mu = 500$ (old score) $H_A: \mu < 500$ (linear)

$$T = \frac{\bar{X}-\mu}{SE} = \frac{\bar{X}-\mu}{s/\sqrt{n}}$$

p-val $P[T < \text{observed} \mid H_0] = P\left[T < \underbrace{\frac{480-520}{80/\sqrt{500}}}_{-\sqrt{500}/4 \doteq 6} \,\middle|\, H_0\right]$



p-val << 0.001 → so evidence suggests H_0 is not so.

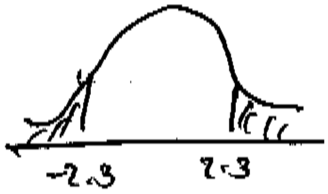
10-15 pop is exp(λ)

$H_0: \lambda = 1$ $H_A: \lambda \neq 1$.

what test statistic? Book suggests using fact that $\bar{X}$ is approx. normal because n is large (Did simulation in lab)

so $T = \dfrac{\bar{X}-\mu}{s/\sqrt{n}}$ would work if we knew s. However in this case we know σ! Why? Pop is exp(λ) is characterized by λ $\left[\mu = \frac{1}{\lambda}, \sigma = \frac{1}{\lambda}\right]$ so we have

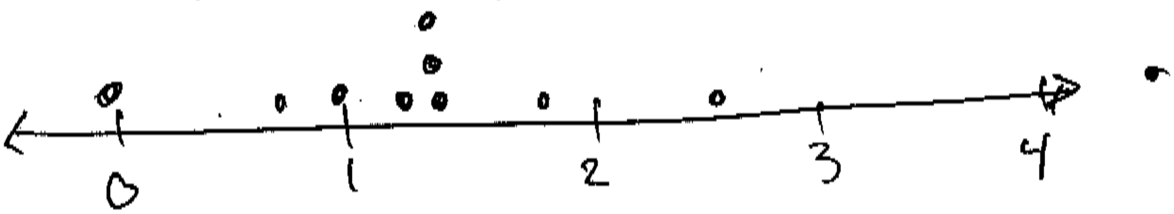
$$T\text{-obs} = \frac{\bar{X}-1}{1/\sqrt{n}} = \frac{1.23-1}{1/\sqrt{100}} = 2.3$$

p-val = P[T more extreme than 2.3] =  < .05 using $z = \pm 2$.

10-18 $H_0: \mu = 0$ $H_A: \mu \neq 0$

Data 1.2, 2.4, 1.3, 1.3, 0, 1.0, 1.8, 0.8, 4.6, 1.3

Dotplot.



Guess* $\bar{x} = 1.5$ $S = \frac{\text{Range}}{6} = \frac{4.6}{6} = .8$ say.

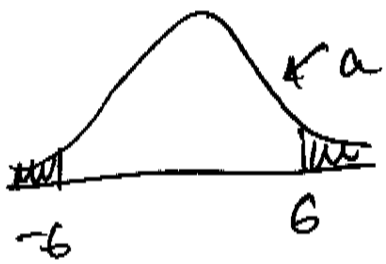
*No calculator is handy.

Data appears to come from $N(\mu, \sigma^2)$ population — use t test.

p-val = P[T more extreme than observed | H_0]

observed $= \dfrac{\bar{X} - \mu}{s/\sqrt{n}} = \dfrac{1.5 - 0}{.8/\sqrt{10}} \approx \sqrt{10} \cdot 2 \approx 6$

p-val. :



← a t_9 distribution ($n = 10$).

p-val is small << .001.

10-20. $H_0: \mu = 0$ $H_A: \mu \neq 0$

assume normality of population.

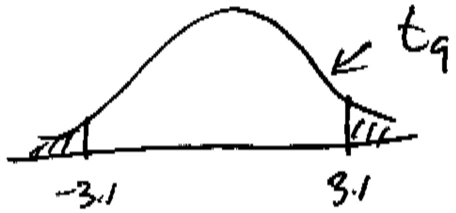
Data $n = 10$, $\bar{x} = 10$ $S = 10.6$ use t-test

10-20 (cont).

$$T\text{-obs} = \frac{\bar{X}-\mu}{s/\sqrt{n}} = \frac{10-0}{10.6/\sqrt{10}} \approx 1\cdot\sqrt{10} \sim 3.1.$$

$$p\text{-val} = P[T > T\text{obs or } T < -T.\text{obs} \mid H_0]$$

$=$



$$p\text{-val} \approx .002$$

[out of order, 2,5,6 on p ~~6~~ 7]

11-8. $H_0: p = \frac{1}{4}$ $H_A: p > \frac{1}{4}$.

Rule reject if $X \geq 7$ $X \sim \text{Binomial}(n=10, p=p)$.

$$\alpha = P[X \geq 7 \mid H_0] = P[X \geq 7 \mid p = \tfrac{1}{4}]$$

$$= \sum_{k=7}^{10} \binom{10}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{10-k} = (1-\text{pbinom}(6, n=10, p=\tfrac{1}{4}))$$

$$\beta(p) = P[\text{Accept} \mid p] = P[X < 7 \mid p]$$

$$= \sum_{k=0}^{6} \binom{10}{k} p^k (1-p)^{10-k} = \text{pbinom}(6, n=10, p).$$

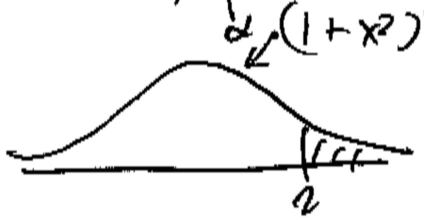
$$\text{power} = 1 - \beta(p)$$

11-11 test $\theta = 0$ on basis of single observation.

$$f(x|\theta) = \frac{1}{\pi} \frac{1}{1+(x-\theta)^2}.$$

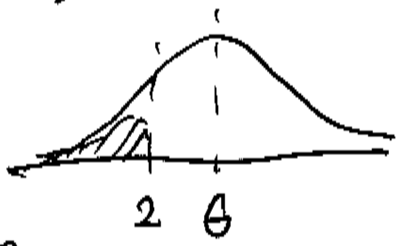
a) if $R = \{X > 2\}$ find α, $\beta(\theta)$.

$\alpha = P[X > 2 | \theta = 0] =$



$$= \int_2^\infty \frac{1}{\pi} \frac{1}{1+x^2}\, dx = -\frac{1}{\pi} \arctan 2 + \frac{\pi}{2}.$$

$\beta = P[X < 2 | \theta] =$

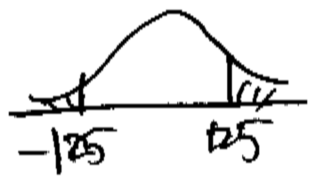


$$= \int_{-\infty}^{2} \frac{1}{\pi} \cdot \frac{1}{1+(x-\theta)^2}\, dx = \int_{-\infty}^{2-\theta} \frac{1}{\pi} \frac{1}{1+x^2}\, dx = \arctan$$

$$= \arctan(2-\theta) - \left(-\frac{\pi}{2}\right)$$

b) if $R = \{|X| > 2\}$ then 2-sided picture.

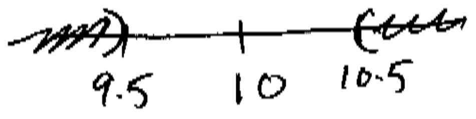
$\alpha = P[\,|X| > 2 \,|\, H_0] =$



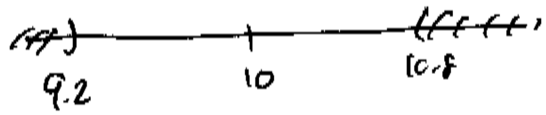
$$= 2 \times \int_{1.5}^{\infty} \frac{1}{\pi} \cdot \frac{1}{1+x^2}\, dx$$

$$\beta = P(|X| < 2 | \theta) = \int_{-2}^{2} \frac{1}{\pi} \frac{1}{1+(x-\theta)^2}\, dx.$$

11-2 $R_1 = \{|\bar{X} - 10| > .5\}$



$R_2 = \{|\bar{X} - 10| > .8\}$



$\alpha = P[\text{Reject} \mid H_0]$ bigger if R bigger $\alpha_1 > \alpha_2$

$\beta = P[\text{Accept} \mid H_1]$ smaller if R bigger $\beta_2 < \beta_1$

11-5 Reject $H_0: \mu = 10$ if 90% CI for μ does not include $\mu = 10$. (assume ~~a~~ $N(\mu, 1)$ population)

a) write as a rejection rule for $\bar{x}$.

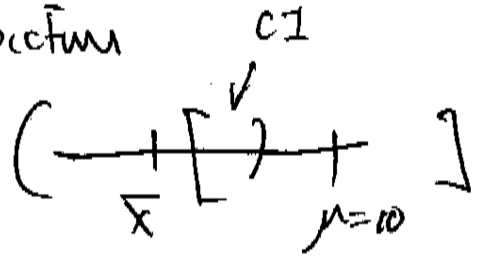
The 90% CI is based on $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ having $N(0,1)$ dist

So $.90 = P\left[\left|\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right| \le z^*\right]$ here $z^* = 1.65$.

Reject if 10 not in $\bar{X} \pm 1.65 \dfrac{1}{\sqrt{n}}$

or reversing $\bar{X}$ not in $\mu \pm 1.65 \dfrac{1}{\sqrt{n}}$.

picture



if CI around $\bar{x}$ misses μ then same size interval around μ misses $\bar{x}$.

$\alpha = P[\text{CI misses } \mu] = P\left[\left|\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right| > z^*\right] = 1 - .90$

11-3 (CONT)

This problem is about the relationship between p-vals. and CIs:

If a $(1-\alpha)100\%$ CI contains μ_0 then the p-val of two sided test $H_0: \mu = \mu_0$ has p-val $\geq \alpha$ (and vice versa).

11-6

$P[$no effect whatsoever$]$

$= P[\cancel{\text{Accept}}\ \text{Reject} \mid H_0]$

both tests

by independence $= P[\text{Reject test } 1 \mid H_0] \times P[\text{Reject test } 2 \mid H_0]$

$= \alpha^2 = 0.05^2 = .0025$

If we want this to be .01 take $\alpha = \sqrt{.01} = .10$.

11.18 $H_0: \mu = 0$ $H_A: \mu = 1$ pop $N(\mu, 1)$.

By Neyman Pearson

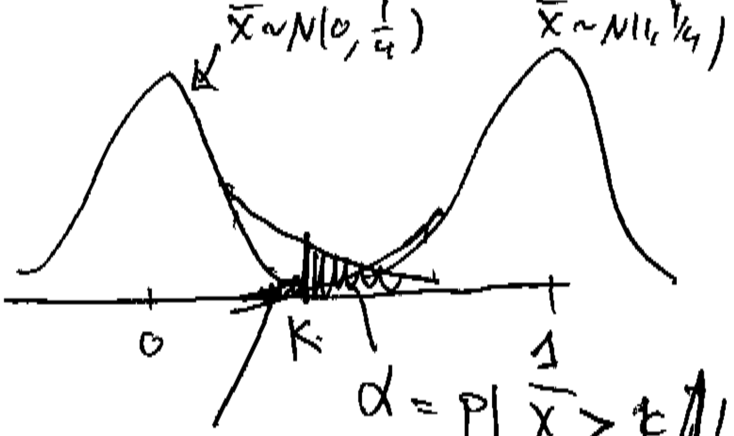
$$\Lambda = \frac{L(0)}{L(1)} < k \text{ is most power}$$

$$L(\mu) = e^{-\frac{1}{2}n(\bar{x}-\mu)^2}$$

so $\dfrac{L(0)}{L(1)} = \dfrac{e^{\cdots}}{e^{-\frac{1}{2}n(\bar{X}-1)^2}}$

and $\Lambda < k$ becomes $\bar{X} > k$ by algebra.

b) when $n=4$ find α, β:



Solving

$\alpha = 1 - \Phi\left(\dfrac{k-0}{1/2}\right)$

$\beta = \Phi\left[\dfrac{k-1}{1/2}\right]$.

$\alpha = P[\bar{X} > k \mid H_0]$

$\beta = P[\bar{X} < k \mid H_1]$.

c) sketch.

11-21. For poisson(m) $L(m) = \prod_{i=1}^{n} \dfrac{m^{x_i}}{x_i!} e^{-m}$

$\propto m^{n\bar{x}} e^{-nm}$

So $\Lambda < k$ becomes $\dfrac{1^{n\bar{x}} e^{-n \cdot 1}}{2^{n\bar{x}} e^{-n \cdot 2}} = \left(\frac{1}{2}\right)^{n\bar{x}} e^{+n}$

So $\Lambda < k \Rightarrow \left(\frac{1}{2}\right)^{n\bar{x}} < k \quad n\bar{x} < \log_{1/2} k = k'$

$\underline{\bar{x} < k''}$

11-25 $H_0: \lambda=\lambda_0$ $H_A: \lambda\neq\lambda_0$

$$\Lambda = \frac{\sup_{\Theta_0} L(\lambda)}{\sup_{\Theta_0\cup\Theta_A} L(\lambda)} = \frac{L(\hat\lambda_0)}{L(\hat\lambda)} = \frac{L(\lambda_0)}{L(\frac{1}{\bar X})} = \frac{\lambda_0^{n\bar x}\, e^{-\lambda_0 n\bar x}}{(\frac{1}{\bar x})^{n\bar x}\, e^{-\frac{1}{\bar x}n\bar x}}$$

$$= [\lambda_0\bar X]^{nx}\, e^{-n[\lambda_0\bar x]} \cdot e^{n}$$

because
$\hat\lambda_0=\lambda_0$ only point

$\hat\lambda$ came from.

① $L(\lambda)=\prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda n\bar x}$

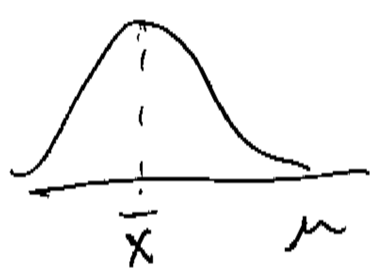
$-\log L(\lambda) = -n\log\lambda + \lambda n\bar x$

$(-\log L(\lambda))'=0$ when $\hat\lambda=\frac{1}{\bar x}$.

11-28 $L(\mu)=e^{-\frac{n}{2}(\bar X-\mu)^2}$

Sketch $L(\mu)$ if $\bar X<0$, if $\bar X>0$.

Fact $L(\mu)$ peaks at $\bar X$!.



so if $\bar X<0$ $\max_{\mu\le 0}$ is at $\bar X$

if $\bar X>0$ $\max_{\mu\le 0}$ at $\mu=0$

$$\text{so } \Lambda = \frac{\sup_{\mu\le 0} L(\mu)}{\sup L(\mu)} = \begin{cases} \frac{L(\bar X)}{L(\bar X)} & \text{if } \bar X<0 \\ \frac{L(0)}{L(\bar X)} & \text{if } \bar X>0. \end{cases}$$

c) show $\Lambda < K$ same as $\bar{X} > c$.

we have $\Lambda = \cancel{\frac{L(0)}{L(\bar{x})}}$ is small only if $\bar{X} > 0$.

and thus $\Lambda < K$ same as $\dfrac{e^{-\frac{n}{2}(\bar{x})^2}}{e^{-\frac{n}{2}(\bar{x}-\mu)^2}} < K$

see now 11-18.

12-13
compare the following

```
> x = c(62,66,90,92, ... , 88,62)
> y = c(64,58, ... ,74,66)
> t-test(x,y)
```

$[H_0: \mu_1 = \mu_2 \quad H_A: \mu_1 \neq \mu_2]$

```
> wilcox.test(x,y) .
```

~~for latter we can~~

Two-sided alternative.

12-14. $H_0: \mu_1 = \mu_2 \quad H_A: \mu_1 \neq \mu_2$.

Assume 1) Normal pop
2) $\sigma_1 = \sigma_2$

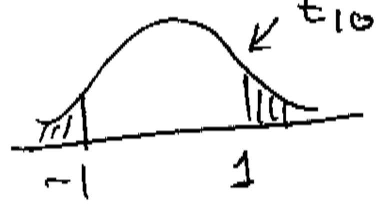
Then $T = \dfrac{\bar{X}_1 - \bar{X}_2}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$ is test statistic with n_1+n_2-2 df.

$\cdot$ T-observed $= \dfrac{\quad}{S_p \sqrt{\frac{1}{8}+\frac{1}{4}}} \approx \dfrac{\quad}{4.7\sqrt{\frac{3}{8}}} = -1$

where $S_p^2 = \dfrac{7 \cdot 4.18^2 + 3 \cdot 5.06^2}{8+4-2} \sim (4.7)^2$ say.

So p-value $=$ P[T > observed or T < -observed | H_0]

$=$

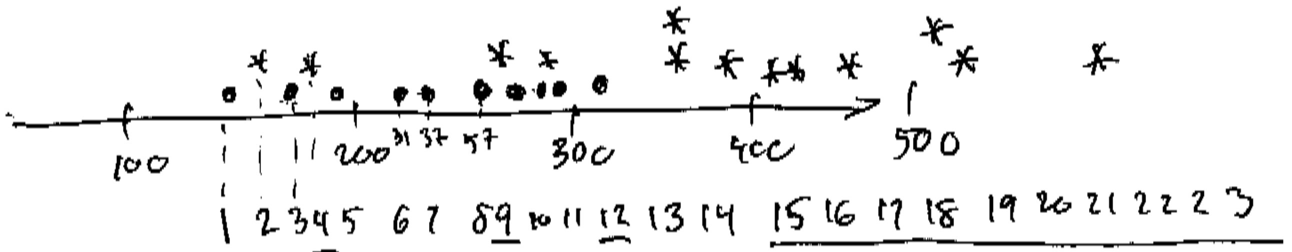


not small

12-16.

x = c(257, 185, 231, 220, 141, 237, 199, 295, 276, 319)
y = c(597, ...)
wilcox.test(x, y)

or



$T = 1+3+5+6+7+8+10+11+13+14$
$= 78.$

$H_0: M_1 = M_2 \quad H_A: M_1 < M_2$

p-val P[T < obs | H_0]

$Z = \dfrac{\text{obs} - \frac{1}{2} - \frac{1}{2} n_1(n_1+n_2+1)}{\sqrt{\frac{1}{12} n_1 n_2 (n_1+n_2+1)}} = \dfrac{78 - \frac{1}{2} - \frac{1}{2}\cdot 10\,[24]}{\sqrt{\frac{1}{12}\, 10 \cdot 14 \cdot 25}}$

observed Z like -3