1. Compute the following integrals:

(a) $\displaystyle\int \frac{\cos x}{1-\sin x}\,dx$

Solution: Set $u = 1-\sin x$, substitute with $du = -\cos x\,dx$ to get

$$\int \frac{\cos x}{1-\sin x}\,dx = -\int \frac{1}{u}\,du = -\ln|1-\sin x| + C.$$

(b) $\displaystyle\int \frac{x}{\sqrt{1-2x^2}}\,dx$

Solution: Set $u = 1-2x^2$, $du = -4x\,dx$, substitute:

$$\int \frac{x}{\sqrt{1-2x^2}}\,dx = -\frac{1}{4}\frac{du}{\sqrt{u}} = -\frac{1}{4}\frac{u^{1/2}}{1/2} = -\frac{1}{2}\sqrt{1-2x^2} + C.$$

(c) $\displaystyle\int_{-1}^{0} t(t+1)^{11}\,dt$

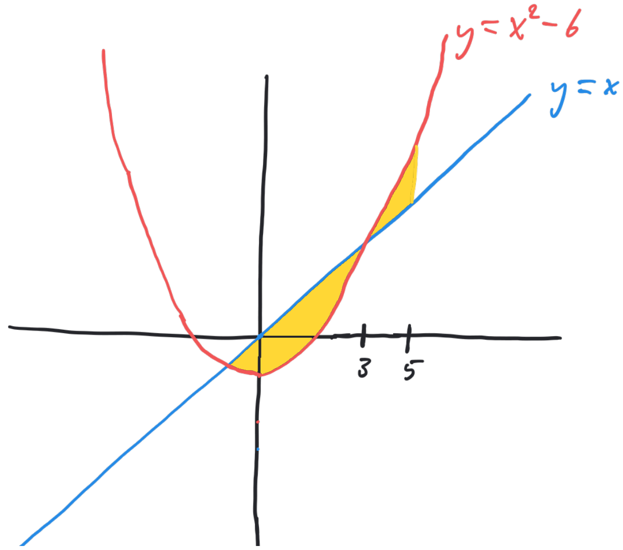
Solution: Set $u = t+1$, $du = dt$, substitute:

$$\int_{-1}^{0} t(t+1)^{11}\,dt = \int_0^1 (u-1)u^{11}\,du = \int_0^1 (u^{12}-u^{11})\,du = \frac{u^{13}}{13} - \frac{u^{12}}{u}\Big|_0^1$$
$$= \frac{1}{13} - \frac{1}{12} = -\frac{1}{156}.$$

The key thing here is that you have to use the substitution $t = u-1$ in the first step.

2. Find the area bounded between the curves $y = x$ and $y = x^2 - 6$ from $x = 0$ to $x = 5$.

Solution: The graphs look like this:



We need to figure out where the two curves intersect. Setting $x = x^2 - 6$ and solving, we get $x = -2, 3$. So, the curve $y = x$ is above $y = x^2 - 6$ from 0 to 3 but below from 3 to 5. The area between the two curves is then

$$\int_0^3 \big(x - (x^2 - 6)\big)\, dx + \int_3^5 \big((x^2 - 6) - x\big)\, dx = 157/6.$$

3. Consider the region bounded by $y = x^2 - 4x$ and the x-axis.
   (a) What's the volume of this region rotated around the x-axis?
   (b) What's the volume of this region rotated around the axis $y = 1$?

Solution: The zeros of $x^2 - 4x$ are $x = 0, 4$. The vertical cross-sections of the region rotated around the x-axis are circles. The slightly nonstandard thing in this problem is that $y = x^2 - 4x$ is *below* the x-axis. So, each cross-section is a circle with radius $-(x^2 - 4x)$. Thus the region has volume

$$\int_0^4 \pi\Big(-(x^2 - 4x)\Big)^2\, dx = \int_0^4 (x^4 - 8x^3 + 16x^2)\, dx = \frac{512\pi}{15}.$$

When we rotate around $y = 1$, each cross-section is an annulus with outer radius $1 - (x^2 - 4x)$ and inner radius 1. Thus the volume is

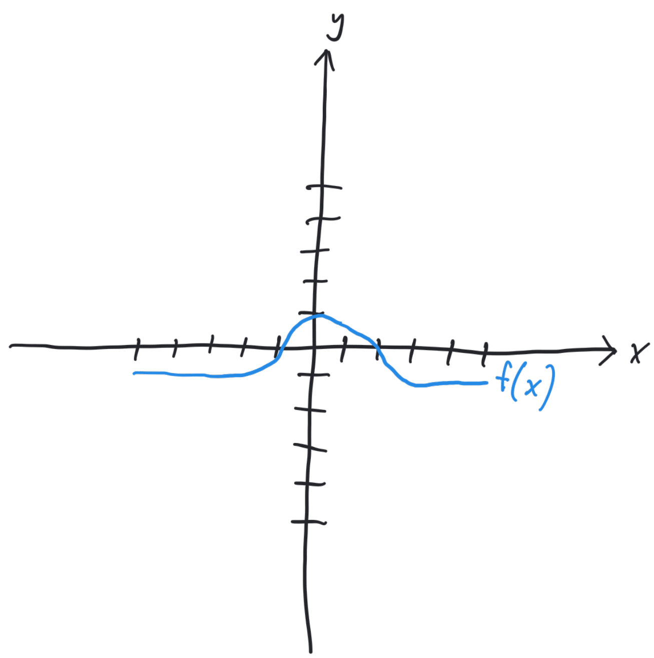
$$\int_0^4 \pi\Big(\big(1 - (x^2 - 4x)\big)^2 - 1^2\Big)\, dx = \frac{832\pi}{15}.$$

4. Consider the pyramid whose base is a 1×1 square and whose top vertex is at height 1 above the square. Find its volume using an integral.

Solution: Using similar triangles, the horizontal cross-section at distance h from the top of the pyramid is a square with side length h. (This is the hard part, but if you draw a picture you'll see it.) So, the area of this cross-section is h^2. We add up all of these areas as h ranges from 0 to 1 to find the volume:

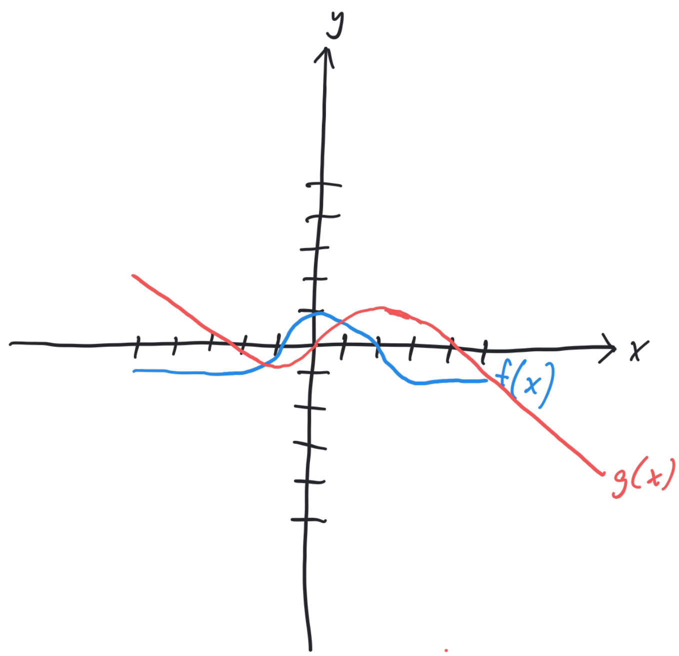
$$\int_0^1 h^2 \, dh = \frac{1}{3}.$$

5. Consider the graph of f shown below:



Let $g(x) = \int_0^x f(t) \, dt$.
Find $g(0)$, $g'(0)$, and $g'(2)$. Sketch $g(x)$.

Solution: We have $g(0)$ since it's an integral from 0 to 0. For $g'(0)$ and $g'(2)$, we use the Fundamental Theorem of Calculus, which says that $g'(x) = f(x)$. So $g'(0) = 1$ and $g'(2) = 0$. Here's a sketch of g. The idea is to draw a function whose derivative is f and that goes through the origin (since $g(0) = 0$).



6. The concentration of live virus in a petri dish declines exponentially over the course of a day. After t hours, the concentration of virus is e^{-3t} times its original level.

   What is the average level of virus throughout the day (i.e., from time $t = 0$ to $t = 24$)?

Solution: It's given by

$$\frac{1}{24}\int_0^{24} e^{-3t}\,dt = \frac{1}{24}\left(-\frac{1}{3}e^{-3t}\right)\Big|_0^{24} = \frac{1}{72}(1-e^{-72}) \approx \frac{1}{72}$$

That is, on average the amount of virus throughout the day is about 1/72 times its original level.