1. Do the following series converge? Explain your answer, and give the full details of any test to use. (For example, for the comparison test, say what series you're comparing to. For the ratio or root test, give the value of the limit you compute when applying the test.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^n}$

Solution: Use the root test.

$$\lim_{n\to\infty} \left(\frac{5^n}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{5}{n} = 0.$$

Since this is less than 1, the series converges.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

Solution: This converges by the alternating series test, since $1/\ln n$ is decreasing and converges to 0.

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\sqrt{n^2+2}$

Solution: This series diverges, since $(-1)^{n+1}\sqrt{n^2+2}$ does not converge to 0 as $n \to \infty$ (it doesn't converge at all).

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4n}$

Solution: The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test. Since $\frac{1}{n^2+4n} \leq \frac{1}{n^2}$, by the comparison test our series converges too.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{2n+3}$

Solution: This series diverges because $\frac{n-2}{2n+3}$ doesn't converge to 0 as $n \to \infty$ (it converges to 1/2).

2. For which values of x does the following power series converge?

$$\sum_{n=1}^{\infty} \frac{n^2}{5^n}(x-8)^n$$

Solution: The ratio of the $(n+1)$th and nth terms of the power series is

$$\frac{\frac{(n+1)^2}{5^{n+1}}(x-8)^{n+1}}{\frac{n^2}{5^n}(x-8)^n} = \left(\frac{n+1}{n}\right)^2 \frac{x-8}{5}$$

The limit of the absolute value of this as $n \to \infty$ is

$$\left|\frac{x-8}{5}\right|,$$

since

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = \lim_{n\to\infty}\left(\frac{1+1/n}{1}\right)^2 = 1^2 = 1.$$

By the ratio test, the series converges when $|\frac{x-8}{5}| < 1$ and diverges when $|\frac{x-8}{5}| > 1$. The condition $|\frac{x-8}{5}| < 1$ holds when $3 < x < 13$. So we get convergence for x in this interval and divergence for $x < 3$ and $x > 13$.

We still need to figure out what happens at $x = 3$ and $x = 13$. For $x = 3$, the series is

$$\sum_{n=1}^{\infty}(-1)^n n^2,$$

and when $x = 13$, the series is

$$\sum_{n=1}^{\infty} n^2.$$

Both of these series diverge since their nth terms don't converge to 0.

So, the series converges for x in the interval $(3, 13)$.

3. Consider the function $f(x) = \sqrt{4+x}$. Its first four derivatives are:

$$\begin{aligned}
f'(x) &= \frac{1}{2(4+x)^{1/2}}, \\
f''(x) &= -\frac{1}{4(4+x)^{3/2}}, \\
f^{(3)}(x) &= \frac{3}{8(4+x)^{5/2}}, \\
f^{(4)}(x) &= -\frac{15}{16(4+x)^{7/2}},
\end{aligned}$$

and the degree 3 Taylor polynomial $f(x)$ centered at 5 is

$$T_3(x) = 3 + \frac{1}{6}(x-5) - \frac{1}{216}(x-5)^2 + \frac{1}{3888(x-5)^3}.$$

According to the Taylor polynomial error bound, what is the largest $|f(5.9) - T_3(5.9)|$ could be?

Solution: To apply the error bound, we need to find a value for K so that $|f^{(4)}(x)| \leq K$ for $5 < x < 5.9$. Since $|f^{(4)}(x)|$ is decreasing,

$$|f^{(4)}(x)| \leq |f^{(4)}(5)| = \frac{5}{11664}.$$

Now the error bound gives us

$$|f(5.9) - T_3(5.9)| \leq \frac{K|5.9-5|^4}{4!}$$

where $K = \frac{5}{11664}$.

4. Let $f(x) = 2x^{1/5}$.

(a) Compute $f'(x)$, $f''(x)$, and $f'''(x)$.

Solution:

$$f'(x) = \frac{2}{5}x^{-4/5},$$
$$f''(x) = -\frac{8}{25}x^{-9/5}$$
$$f'''(x) = \frac{72}{125}x^{-14/5}.$$

(b) What is the 3rd degree Taylor polynomial for $f(x)$ centered at 1?

Solution: We compute

$$f(1) = 2,$$
$$f'(1) = \frac{2}{5},$$
$$f''(1) = -\frac{8}{25}$$
$$f'''(1) = \frac{72}{125}$$

The Taylor polynomial is

$$f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3$$
$$=2+\frac{2}{5}(x-1)-\frac{4}{25}(x-1)^2+\frac{12}{125}(x-1)^3.$$

5. Find the 4th degree Taylor polynomial for $\ln(x)$ centered at 2.

Solution: Let $f(x)=\ln x$.

$$f'(x)=x^{-1}, \qquad \frac{f'(x)}{1!}=\frac{1}{x}$$
$$f''(x)=-x^{-2}, \qquad \frac{f''(x)}{2!}=-\frac{1}{2x^2}$$
$$f^{(3)}(x)=2x^{-3}, \qquad \frac{f^{(3)}}{3!}=\frac{1}{3x^3}$$
$$f^{(4)}(x)=-6x^{-4}, \qquad \frac{f^{(4)}}{4!}=\frac{1}{4x^4}$$

In the right column here, I'm putting the nth derivative divided by $n!$ because there's a nice pattern, and next we will want to find the Taylor polynomial coefficients $f^{(n)}(c)/n!$:

$$f(2)=\ln 2,$$
$$\frac{f'(2)}{1!}=\frac{1}{2}$$
$$\frac{f''(2)}{2!}=-\frac{1}{2(2^2)}=-\frac{1}{8},$$
$$\frac{f^{(3)}(2)}{3!}=\frac{1}{3(2^3)}=\frac{1}{24},$$
$$\frac{f^{(4)}(2)}{4!}=-\frac{1}{4(2^4)}=-\frac{1}{64}.$$

And so the Taylor polynomial is

$$\ln 2+\frac{1}{2}(x-2)-\frac{1}{8}(x-2)^2+\frac{1}{24}(x-2)^3-\frac{1}{64}(x-2)^4.$$

6. Find the Taylor series for $f(x)=1/x$ centered at -3.

Solution: Find

$$\begin{aligned} f'(x) &= -x^{-2}, \\ f''(x) &= 2x^{-3}, \\ f^{(3)}(x) &= -(3)(2)x^{-4}, \\ f^{(4)}(x) &= (4)(3)(2)x^{-5}. \end{aligned}$$

Each time we take another derivative, we multiply by a number that one higher than the last, and we negate the coefficient, and we decrease the exponent of x by 1. So,

$$f^{(n)}(x) = n!(-1)^n x^{-(n+1)}.$$

So th Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x+3)^n = \sum_{n=0}^{\infty} \frac{n!(-1)^n(-3)^{-(n+1)}}{n!}(x+3)^n = \sum_{n=0}^{\infty} -\frac{(x+3)^n}{3^{n+1}}$$

7. How accurate is the following estimate?

$$\cos(0.3) \approx 1 - \frac{(0.3)^2}{2!} + \frac{(0.3)^4}{4!}$$

To answer the question, provide a bound on how far the right-hand side is from the left-hand side using the Taylor polynomial error bound.

Solution: The expression approximating $\cos(0.3)$ is the degree 5 Taylor polynomial for $\cos x$ centered at 0. (It's also the degree 4 Taylor polynomial.) According to the Taylor polynomial error bound,

$$|\cos(x) - T_5(x)| \leq \frac{K|x|^6}{6!},$$

where K is a constant such that

$$\left| \frac{d^6}{du^6} \cos u \right| \leq K$$

for all u between 0 and x. We are going to apply this error bound with $x = 0.3$. For the upper bound on the 6th derivative of $\cos x$, we first see that the 6th derivative of $\cos x$ is $-\cos x$. It's always true that $|\cos x| \leq 1$, and we can't do any better than this for an upper bound since $\cos 0 = 1$, so we'll use $K = 1$. Now the error bound gives

$$|\cos(0.3) - T_5(0.3)| \leq \frac{(0.3)^6}{6!} = .0000010125.$$

8. For which values of x do the following sequences converge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n} x^n$

Solution: Apply the ratio test. The ratio of the $(n+1)$th and nth terms of the series is

$$\frac{\frac{(n+1)^2}{2^{n+1}} x^{n+1}}{\frac{n^2}{2^n} x^n} = \left(\frac{n+1}{n}\right)^2 \frac{x}{2}$$

We take the limit of the absolute value of this ratio:

$$\begin{aligned}\lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)^2 \frac{x}{2} \right| &= \frac{|x|}{2} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \\ &= \frac{|x|}{2} \lim_{n\to\infty} \left(\frac{1+1/n}{1}\right)^2 = \frac{|x|}{2}.\end{aligned}$$

The ratio test says that the series converges when $|x|/2 < 1$, i.e., when $-2 < x < 2$, and that it diverges when $x < -2$ or $x > 2$.

Now we have to check what happens at $x = -2$ and $x = 2$. At $x = -2$, the series is

$$\sum_{n=1}^{\infty} (-1)^n n^2,$$

which diverges because $(-1)^n n^2$ does not converge to 0 as $n \to \infty$ (it doesn't converge at all). At $x = 2$, the series is

$$\sum_{n=1}^{\infty} n^2,$$

which again diverges for the same reason.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\pi^n x^n}{n^\pi}$

Solution: As usual, we want to use the ratio test. The ratio of the $(n+1)$th and nth terms of the series is

$$\frac{\frac{\pi^{n+1} x^{n+1}}{(n+1)^\pi}}{\frac{\pi^n x^n}{n^\pi}} = \pi x \left(\frac{n}{n+1}\right)^\pi$$

The limit of the absolute value of this ratio is

$$\lim_{n\to\infty} \pi x \left(\frac{n}{n+1}\right)^\pi = \pi|x| \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^\pi$$
$$= \pi|x| \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^\pi = \pi|x|.$$

By the ratio test, the series converges when $\pi|x| < 1$, i.e., when $-\frac{1}{\pi} < x < \frac{1}{\pi}$. As in the last problem, we need to check what happens when x is on the boundary of this interval. At $x = \frac{1}{\pi}$, the series is

$$\sum_{n=1}^\infty \frac{1}{n^\pi}.$$

This series converges by the p-test, since $\pi > 1$. At $x = -\frac{1}{\pi}$, the series is

$$\sum_{n=1}^\infty \frac{(-1)^n}{n^\pi}.$$

This series is convergent too, since it's absolutely convergent (when we take absolute values of each term we get the series we just examined). Or you could say that it's convergent by the alternating series test.

So, the interval of convergence is the closed interval $[-1/\pi, 1/\pi]$.

9. Is the following statement true or false? If false, explain what is incorrect about it.

The series $\sum_{n=0}^\infty a_n$ converges if $\lim_{n\to\infty} a_n = 0$ and diverges otherwise.

Solution: It's half true. The half that's true is that the series diverges if a_n does not converge to 0. But it is *not* true that the series converges whenever a_n converges to 0.

10. Find the Taylor series for the following function (you can choose the center): $f(x) = xe^{-x^2}$.

Solution: It's perfectly possible to compute the series by taking derivatives, but it's much less work to get it by manipulating known Taylor series. We start with

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Now we substitute x^2 for x:

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

Now we multiply each term by x:

$$xe^{x^2} = x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}.$$

The question doesn't ask anything about the interval of convergence, but because the first series converges everywhere, so do the other two series.

11. Let $f(x) = 1/(1+x^2)$.

(a) What is the Taylor series for $f(x)$ centered at 0? What is its interval of convergence?

Solution: Since $1/(1+x^2)$ is the sum of a geometric series with first term 0 and multiplier $-x^2$ (i.e., each term is $-x^2$ times the last),

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

The interval of convergence is $-1 < x < 1$, since the series converges when the multiplier $-x^2$ is between -1 and 1.

(b) Find $f'(x)$.

Solution: Since $f(x) = (1+x^2)^{-1}$,

$$f'(x) = -(1+x^2)^{-2}(2x) = -\frac{2x}{(1+x^2)^2}.$$

(c) Give the Taylor series for $f'(x)$ centered at 0. What is its radius of convergence?

Solution: We can differentiate the power series for $f(x)$ term by term to get the power series for $f'(x)$:

$$f'(x) = -2x + 4x^3 - 6x^5 + 8x^7 - \cdots.$$

The radius of convergence is 1, since that was the radius of convergence for the original series and differentiating term by term doesn't change the radius of convergence.