

I pledge that I have neither given nor received unauthorized assistance during this examination.

Signature:

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- You may use a scientific calculator, but not a graphing calculator or phone.
- It is okay to leave a numerical answer like $\frac{39}{2} - (18 - e^2)$ unsimplified.
- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 7 problems on 7 pages.

Question	Points	Score
1	12	
2	12	
3	8	
4	8	
5	6	
6	8	
7	6	
Total:	60	

Good luck!

1. Determine for which values of x the following power series converge.

[6 points] (a) $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$

Solution: Apply the ratio test. Ratio of $(n + 1)$ th and n th term is

$$\frac{\frac{(2x)^{n+1}}{n+1}}{\frac{(2x)^n}{n}} = 2x \frac{n+1}{n}.$$

The absolute value of the limit is $2|x|$. So the series converges when $2|x| < 1$, or for $-1/2 < x < 1/2$.

At the boundary, the series are

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

when $x = -1/2$ and

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

The first converges by the alternating series test and the second diverges by the p -test. So the interval of convergence is $[-1/2, 1/2)$.

[6 points] (b) $\sum_{n=1}^{\infty} \frac{n^2 x^n}{n!}$

Solution: Apply the ratio test. Ratio of $(n + 1)$ th and n th term is

$$\frac{\frac{(n+1)^2 x^{n+1}}{(n+1)!}}{\frac{n^2 x^n}{n!}} = x \frac{1}{n+1} \left(\frac{n+1}{n} \right)^2.$$

The absolute value of the limit is 0. So the series converges for all x , and the interval of convergence is $(-\infty, \infty)$.

2. State whether the following series converge or diverge. Explain your answer. If you apply a test, you must give all details of the test to get full credit. (For example, for the comparison test, say what series you're comparing to.)

[4 points] (a) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$

Solution: Use the comparison test:

$$\frac{1}{n\sqrt{n+1}} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by the p -test, the series in this problem converges too.

[4 points] (b) $\sum_{n=1}^{\infty} \frac{n-1}{n}$

Solution: Since $\frac{n-1}{n}$ converges to 1 as $n \rightarrow \infty$ and not to 0, this series diverges.

[4 points] (c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Solution: This series converges by the alternating series test.

3. State whether the following series converge or diverge. If they converge, **compute the sum**. If not, explain why they diverge. If you apply a test, you must give all details of the test to get full credit. (For example, for the comparison test, say what series you're comparing to.)

[4 points] (a) $\sum_{n=1}^{\infty} e^{-2n}$

Solution: This is a geometric series with first term e^{-2} and multiplier $e^{-2} < 1$. By the formula for the sum of a geometric series, it sums to

$$\frac{e^{-2}}{1 - e^{-2}} = \frac{1}{e^2 - 1} \approx 0.1565.$$

[4 points] (b) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots$

Solution: This is a geometric series with first term 1 and multiplier $-1/2$. By the formula for the sum of a geometric series, it sums to

$$\frac{1}{1 + 1/2} = \frac{2}{3}.$$

- [8 points] 4. Find the degree 3 Taylor polynomial $T_3(x)$ for the function $f(x) = x^{1/3}$ centered at 1.

Solution: Start by taking derivatives of $f(x)$:

$$\begin{aligned}f'(x) &= \frac{1}{3}x^{-2/3}, \\f''(x) &= -\frac{2}{9}x^{-5/3}, \\f^{(3)}(x) &= \frac{10}{27}x^{-8/3}.\end{aligned}$$

Now evaluate them at 1:

$$\begin{aligned}f(1) &= 1, \\f'(1) &= \frac{1}{3}, \\f''(1) &= -\frac{2}{9}, \\f'''(1) &= \frac{10}{27}.\end{aligned}$$

Now the Taylor polynomial is

$$T_3(x) = 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{5}{81}(x-1)^3.$$

- [6 points] 5. Consider the function $f(x) = e^x$. The degree 2 Taylor polynomial for $f(x)$ centered at 0 is

$$T_2(x) = 1 + x + \frac{1}{2!}x^2.$$

According to the Taylor polynomial error bound, what is the largest $|f(-0.1) - T_2(-0.1)|$ could be?

Solution: To apply the error bound, we need to find a value for K so that $|f^{(3)}(x)| \leq K$ for $-0.1 \leq x \leq 0$. Since $|f^{(3)}(x)| = e^x$ is decreasing, the biggest it gets on that interval is at $x = 0$, when it is equal to 1. So we can take $K = 1$.

Now the error bound gives us

$$|f(-0.1) - T_2(-0.1)| \leq \frac{|-0.1 - 0|^3}{3!} = \frac{0.001}{6} = .000166666\dots$$

6. Find the Taylor series centered at 0 for the following functions:

[4 points]

(a) $f(x) = x^2 \sin(2x^2)$

Solution: Start with $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$. Now substitute $2x^2$ for x and then multiply by x^2 to get

$$\begin{aligned} x^2 \left(2x^2 - \frac{2^3 x^6}{3!} + \frac{2^5 x^{10}}{5!} - \dots \right) &= 2x^4 - \frac{2^3 x^8}{3!} + \frac{2^5 x^{12}}{5!} - \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{4n+4}}{(2n+1)!} \end{aligned}$$

[4 points]

(b) $f(x) = \frac{x^3}{1-x^2}$

Solution: Either start with $1/(1-x) = 1 + x + x^2 + \dots$ and then substitute x^2 for x and multiply by x^3 , or just observe that $\frac{x^3}{1-x^2}$ is the sum of a geometric series with first term x^3 and multiplier x^2 . Both approaches lead to

$$f(x) = x^3 + x^5 + x^7 + \dots = \sum_{n=0}^{\infty} x^{2n+3}.$$

[6 points] 7. Find the Taylor series of $f(x) = (\sin x)^2$ centered at 0.

Hint: $f'(x) = 2 \sin(x) \cos(x) = \sin(2x)$. Or equivalently, $f(x)$ is an antiderivative of $\sin(2x)$.

Solution: We start by writing the Taylor series for $\sin(2x)$, which we get from $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ by substitution:

$$\sin(2x) = 2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

Now we integrate term by term to get

$$(\sin x)^2 + C = x^2 - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+2}}{(2n+2)!}$$

Plugging in $x = 0$, we have C on the left-hand side and 0 on the right-hand side. So $C = 0$ and the right-hand side is the Taylor series for $(\sin x)^2$.

You can also just take the derivatives of $f'(x)$ using the hint, finding that

$$\begin{aligned} f'(x) &= \sin(2x), \\ f''(x) &= 2 \cos(2x), \\ f^{(3)}(x) &= -4 \sin(2x), \\ f^{(4)}(x) &= -8 \cos(2x), \\ f^{(5)}(x) &= 16 \sin(2x), \end{aligned}$$

after which the pattern of sines and cosines starts to wrap around. Evaluating these at 0, all of the sines are equal to 0 and all of the cosines are equal to 1, and we end up getting the same series like this as via integration.