1. Compute the following integrals. If they are improper integrals and don't converge, say so.

(a) $\int \frac{x}{\sqrt{x^2+9}}\,dx$

Solution: This one doesn't need trigonometric substitution. Just do $u = x^2+9$, giving $du = 2x\,dx$:

$$\int \frac{x}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \sqrt{u} + C = \sqrt{x^2+9} + C$$

(b) $\int \frac{x^2+5}{(x+1)^2(x-2)}\,dx$

Solution: This requires partial fractions. Set:

$$\frac{x^2+5}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$

Clear fractions:

$$x^2+5 = A(x+1)(x-2) + B(x-2) + C(x+1)^2$$

Set $x = 2$ to solve for C:

$$9 = 9C,$$

so $C = 1$. Now set $x = -1$ to solve for B:

$$6 = -3B,$$

so $B = -2$. Now set x to anything else (we'll use 0) to solve for A:

$$5 = -2A - 2B + C = -2A + 4 + 1,$$

so $A = 0$.
Now we can integrate:

$$\int \frac{x^2+5}{(x+1)^2(x-2)}\,dx = \int \left(\frac{-2}{(x+1)^2} + \frac{1}{x-2}\right)dx$$
$$= \frac{2}{x+1} + \ln|x-2| + C.$$

(c) $\int_0^1 x^2 \ln(x^3)\,dx$

Solution: Substitute $y = x^3$, giving $dy = 3x^2\,dx$:

$$\int_0^1 x^2 \ln(x^3)\,dx = \frac{1}{3}\int_0^1 \ln(y)\,dy. \tag{1}$$

Now we have to integrate $\ln y$, which is harder than it looks. Do integration by parts with $u = \ln y$ and $dv = dy$. Then $du = \frac{1}{y}\,dy$ and $v = y$, and

$$\begin{aligned}\frac{1}{3}\int_0^1 \ln(y)\,dy &= \frac{1}{3}\left(y\ln y\Big|_0^1 - \int_0^1 y\frac{1}{y}\,dy\right)\\ &= \frac{1}{3}\left(y\ln y\Big|_0^1 - \int_0^1 1\,dy\right) = \tfrac{1}{3}y\ln y\Big|_0^1 - \tfrac{1}{3}.\end{aligned} \tag{2}$$

Now, we actually have an improper integral here! (In fact, we did from the start, because $\ln(x^3) \to -\infty$ as $x \to 0$.) This comes up when we have to compute $y\ln y\Big|_0^1$, since $y\ln y$ is undefined at 0. Instead, we have to find $\lim_{R\to 0^+} y\ln y\Big|_R^1$, which is

$$1\ln(1) - \lim_{R\to 0^+} R\ln R.$$

The first term here is 0. The second is also 0, by L'Hopital's rule:

$$\lim_{R\to 0^+} R\ln R = \lim_{R\to 0^+}\frac{\ln R}{1/R} = \lim_{R\to 0^+}\frac{1/R}{-1/R^2} = \lim_{R\to 0^+}(-R) = 0.$$

Going back to (2) and then (1), we've computed the integral to be equal to $-\frac{1}{3}$.

(d) $\displaystyle\int_0^\infty \frac{1}{16x^2+1}\,dx$

Solution: You can either just know that the antiderivative is $\frac{1}{4}\tan^{-1}(4x)$, or you can substitute $u = 4x$, with $du = 4\,dx$, to get

$$\int \frac{1}{16x^2+1}\,dx = \frac{1}{4}\int \frac{1}{u^2+1}\,du = \frac{1}{4}\tan^{-1}(u) + C = \frac{1}{4}\tan^{-1}(4x) + C.$$

Now, we compute the improper integral:

$$\begin{aligned}\int_0^\infty \frac{1}{16x^2+1}\,dx &= \lim_{R\to\infty}\int_0^R \frac{1}{16x^2+1}\,dx\\ &= \lim_{R\to\infty}\left(\frac{1}{4}\tan^{-1}(4R) - \frac{1}{4}\tan^{-1}(0)\right).\end{aligned}$$

Now we need to understand the $\tan^{-1}(x)$ function. First, $\tan^{-1}(0) = 0$, since the solution to $\tan\theta = 0$ is $\theta = 0$. Next, what happens to $\tan^{-1}(x)$ as $x \to \infty$? Well, if we are solving $\tan\theta = x$ and x is growing to infinity, then we're looking for an angle so that the y-coordinate divided by the x-coordinate on the unity circle is converging to infinity. This happens when $\theta \to \pi/2$, since as θ inches up to 90 degrees, the y-coordinae vanishes while the x-coordinate approaches 1. So, $\tan^{-1}(x)$ approaches $\pi/2$ as $x \to \infty$. This gives

$$\frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{8}$$

for the integral.

(e) $\displaystyle\int \frac{1}{(11 - x^2)^{1/2}}\, dx$

Solution: Substitute $x = \sqrt{11}\sin\theta$. Then $dx = \sqrt{11}\cos\theta\, d\theta$, and

$$\int \frac{1}{(11 - x^2)^{1/2}}\, dx = \int \frac{\sqrt{11}\cos\theta}{\sqrt{11}\cos\theta}\, d\theta$$
$$= \int 1\, d\theta = \theta + C = \sin^{-1}(x/\sqrt{11}) + C.$$

(f) $\displaystyle\int_{-1/2}^{1/2} \frac{1}{(2x+1)^{1/3}}\, dx$

Solution: Apply the substitution $u = 2x + 1$ to get to

$$\int_{-1/2}^{1/2} \frac{1}{(2x+1)^{1/3}}\, dx = \int_0^2 \frac{1}{2u^{1/3}}\, du.$$

This is an improper integral because the integrand goes off to infinity at the left endpoint. But it converges: we have

$$\int_R^2 \frac{1}{2u^{1/3}}\, du = \frac{3u^{2/3}}{4}\bigg|_R^2 = \frac{3(2^{2/3})}{4} - \frac{3(R^{2/3})}{4}.$$

We can find limit as $R \to 0^+$ just by plugging in $R = 0$ and we get

$$\int_0^2 \frac{1}{2u^{1/3}}\, du = \lim_{R\to 0^+} \int_R^2 \frac{1}{2u^{1/3}}\, du = \frac{3(2^{2/3})}{4}$$

2. Find the volume of the sphere of radius R by rotating the semicircle bounded by $x^2+y^2 = R^2$ about the y-axis, using cylindrical shells.

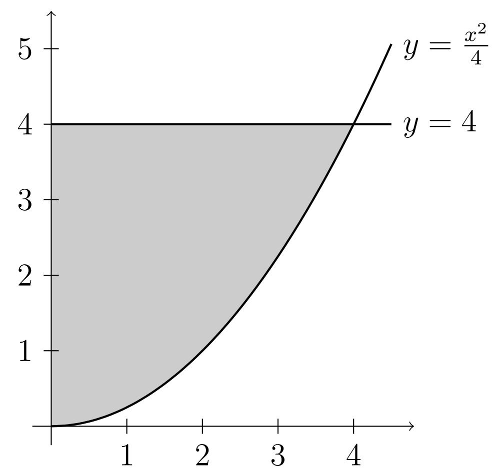
Solution: We integrate with respect to x. The shell at a given x has radius x, and it goes from $y = -\sqrt{R^2 - x^2}$ to $y = \sqrt{R^2 - x^2}$ for a height of $2\sqrt{R^2 - x^2}$. Thus the integral using cylindrical shells is

$$\int_0^R 2\pi x(2\sqrt{R^2 - x^2})\,dx.$$

To compute the integral, use $u = R^2 - x^2$, giving $du = -2x\,dx$, and

$$\int_0^R 2\pi x(2\sqrt{R^2 - x^2})\,dx = -\int_{R^2}^0 2\pi\sqrt{u}\,du = 2\pi\tfrac{2}{3}u^{3/2}\Big|_0^{R^2} = \tfrac{4}{3}\pi R^3.$$

3. Consider the region in between the y-axis, the line $y = 4$, and the curve $y = x^2/4$:



Now revolve this region around the x-axis to form a solid.

Set up *but do not compute* integrals to find the volume of this solid using both the disc/washer method and the cylindrical shells method.

Solution: With washers, we integrate with respect to x. The outer radius of the washer at x is 4 and the inner radius is $x^2/4$. So the integral is

$$\int_0^4 \pi\big(4^2 - (x^2/4)^2\big)\,dx = \pi\int_0^4 \big(16 - x^4/16\big)\,dx.$$

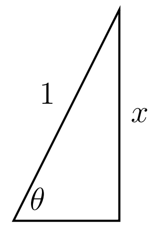
With shells, we integrate with respect to y. We'll need to take the curve $y = x^2/4$ and rewrite it with x in terms of y: $x = 2\sqrt{y}$. The shell at height y has radius y and height $2\sqrt{y}$. The volume is

$$\int_0^4 2\pi y(2\sqrt{y})\,dy = \pi\int_0^4 4y^{3/2}\,dy.$$

4. Here are two questions related to trigonometric substitution:

   (a) Suppose that $\sin\theta = x$. What is $\cot\theta$? (Your answer should be in terms of x but should not involve any inverse trigonometric functions. You can assume that $0 < \theta < \pi/2$.)

   Solution: We make a little triangle based on $\sin\theta = x$:

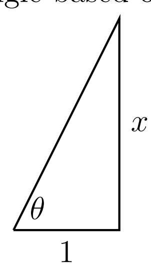
   

   So the other side of the triangle is $\sqrt{1-x^2}$, and

   $$\cot\theta = 1/\tan\theta = \frac{\sqrt{1-x^2}}{x}.$$

   (b) Suppose that $\tan\theta = x$. What is $\csc\theta$? (Your answer should be in terms of x but should not involve any inverse trigonometric functions. You can assume that $0 < \theta < \pi/2$.)

   Solution: We make a little triangle based on $\tan\theta = x$:

   So the other side of the triangle is $\sqrt{1+x^2}$, and

   $$\csc\theta = 1/\sin\theta = \frac{\sqrt{1+x^2}}{x}.$$

5. Compute the following sum, or say that it does not converge.

   $$\sum_{n=2}^{\infty} \frac{3^{n+1}}{5^n}.$$

Solution: This is a geometric series. The first term is $\frac{3^3}{5^2} = \frac{27}{25}$ and at each step we multiply by $\frac{3}{5}$ to get the next term. Using the formula for the sum of a geometric series, it adds up to

$$\frac{27/25}{1-\frac{3}{5}} = \frac{27}{25} \cdot \frac{5}{2} = \frac{27}{10}.$$

6. Does the series

$$\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}$$

converge or diverge? Explain your answer. If you apply a test, you must give all details of the test to get full credit. (For example, for the comparison test, say what series you're comparing to.)

Solution: Recall that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test. We'd like to say that $\frac{n^3}{n^4+1} \geq \frac{n^3}{n^4} = \frac{1}{n}$ so that we can conclude that our series diverges too by the comparison test, but that comparison is incorrect. But the following comparison is correct:

$$\frac{n^3}{n^4+1} \geq \frac{n^3}{2n^4} = \frac{1}{2n}.$$

And $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges (since if it converged, then $\sum_{n=1}^{\infty} \frac{1}{n}$ would converge to twice its value, but we know that it diverges). Hence by the comparison testt our series diverges.

Alternatively, we could use the limit comparison test. We compute

$$\lim_{n\to\infty} \frac{n^3/(n^4+1)}{1/n} = \lim_{n\to\infty} \frac{n^4}{n^4+1} = \lim_{n\to\infty} \frac{1}{1+n^{-4}} = 1.$$

Hence the series $\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ either both converge or both diverge, and we already know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

7. Does the series

$$\sum_{n=2}^{\infty} \frac{n^2-1}{n^2}$$

converge or diverge? Explain your answer. If you apply a test, you must give all details of the test to get full credit. (For example, for the comparison test, say what series you're comparing to.)

Solution: Since $\frac{n^2-1}{n^2}$ converges to 1 as $n \to \infty$ and not to 0, this series diverges.

8. Does the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 4n}$$

converge or diverge? Explain your answer. If you apply a test, you must give all details of the test to get full credit. (For example, for the comparison test, say what series you're comparing to.)

Solution: Recall that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test. Since

$$\frac{1}{n^2 + 4n} \leq \frac{1}{n^2},$$

we get convergence of our sum as well by the comparison test.