

I pledge that I have neither given nor received unauthorized assistance during this examination.

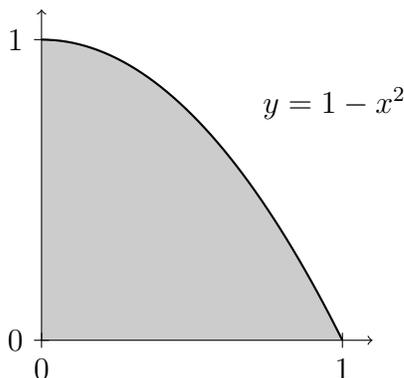
Signature:

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- You may use a scientific calculator, but not a graphing calculator or phone.
- It is okay to leave a numerical answer like $\frac{39}{2} - (18 - e^2)$ unsimplified.
- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 9 problems on 9 pages.

Question	Points	Score
1	8	
2	4	
3	4	
4	6	
5	6	
6	8	
7	8	
8	8	
9	6	
Total:	58	

Good luck!

1. Consider the region underneath the curve $y = 1 - x^2$ from $x = 0$ to $x = 1$, shown here:



Suppose we rotate this region around the y -axis to form a solid.

[4 points]

- (a) Set up **but do not compute** an integral that computes the volume of this region using the disc/washer method.

Solution: For this one, we integrate with respect to y , which means we must write x in terms of y :

$$x = \sqrt{1 - y}.$$

The radius at height y is $x = \sqrt{1 - y}$, and so the volume is

$$\int_0^1 \pi x^2 dy = \int_0^1 \pi(1 - y) dy.$$

[4 points]

- (b) Set up **but do not compute** an integral that computes the volume of this region using the cylindrical shell disc/washer method.

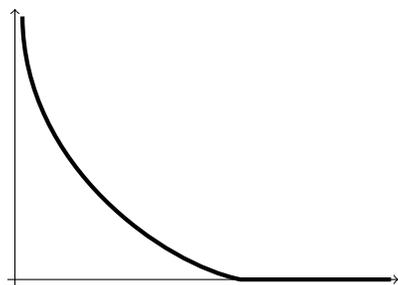
Solution: For this one, we integrate with respect to x :

$$\int_0^1 2\pi x(1 - x^2) dx$$

- [4 points] 2. In each example, determine whether the improper integral $\int_0^\infty f(x) dx$ is convergent based on the sketch.

Assume that the sketch accurately represents the function, i.e., the function continues to follow whatever trend is shown on the sketch. Please note that the functions may have asymptotes both as $x \rightarrow \infty$ and as $x \rightarrow 0$. Answer **yes** if the integral is definitely convergent based on the sketch, **no** if it's definitely not, and **maybe** if the sketch doesn't give enough information to say for sure.

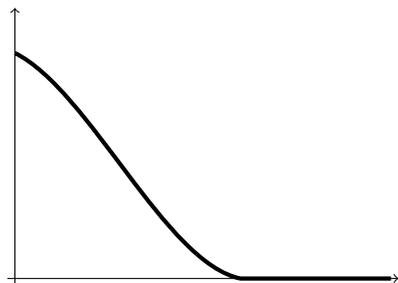
- (a) Is $\int_0^\infty f(x) dx$ convergent for the following function $f(x)$?



Yes No **Maybe**

Solution: The asymptote at 0 might or might not result in a convergent integral.

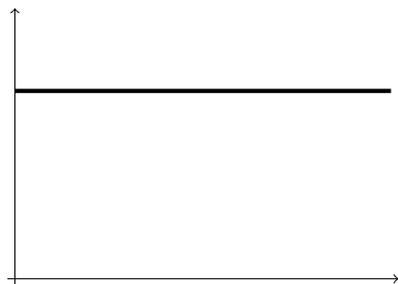
- (b) Is $\int_0^\infty f(x) dx$ convergent for the following function $f(x)$?



Yes No Maybe

Solution: This function is defined at 0 and after a certain point is always 0, so the improper integral must converge.

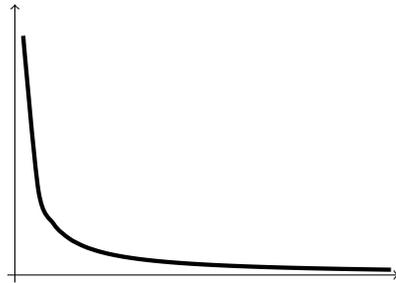
- (c) Is $\int_0^\infty f(x) dx$ convergent for the following function $f(x)$?



Yes **No** Maybe

Solution: This function doesn't converge to 0 as $x \rightarrow \infty$, so there's no hope of integrating it with a right endpoint of ∞ .

(d) Is $\int_0^\infty f(x) dx$ convergent for the following function $f(x)$?



Yes

No

Maybe

Solution: The asymptote at 0 and at ∞ both might cause the improper integral to diverge, or they might not. No way to know.

- [4 points] 3. For the following series, compute the sum or state that it does not converge. Explain your answer. If you apply a test, you must give all details of the test to get full credit. (For example, for the comparison test, say what series you're comparing to.)

$$\sum_{n=1}^{\infty} e^{1/n^2}$$

Solution: As $n \rightarrow \infty$, we have $e^{1/n^2} \rightarrow 1$. Since this limit is to something other than 0, the integral diverges.

- [6 points] 4. Compute $\int_0^{\infty} e^{-3x} dx$ or demonstrate that the integral does not converge.

Solution: First, the antiderivative is

$$\int e^{-3x} dx = -\frac{1}{3}e^{-3x}.$$

Now,

$$\int_0^{\infty} = \lim_{R \rightarrow \infty} \int_0^R \int e^{-3x} dx = \lim_{R \rightarrow \infty} \left(-\frac{1}{3}e^{-3R} + \frac{1}{3} \right) = \frac{1}{3},$$

since $e^{-3R} \rightarrow 0$ as $R \rightarrow \infty$.

5. Consider the series $\sum_{n=1}^{\infty} \frac{3(2^n)}{10^{n-1}}$.

[2 points] (a) List the first three terms of this series.

Solution: They are 6, $\frac{12}{10}$, and $\frac{24}{100}$.

[4 points] (b) Compute the sum or state that it does not converge. Explain your answer. If you apply a test, you must give all details of the test to get full credit. (For example, for the comparison test, say what series you're comparing to.)

Solution: This is a geometric series. The first term is 6 and each successive term is obtained by multiplying by $2/10 = \frac{1}{5}$. By the formula for the sum of a geometric series, the sum is

$$\frac{6}{1 - \frac{1}{5}} = \frac{15}{2}.$$

[8 points] 6. Find $\int \frac{x^2}{\sqrt{4-x^2}} dx$.

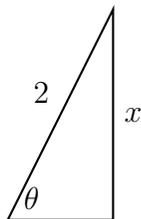
Solution: This calls for the trigonometric substitution $x = 2 \sin \theta$. Then $dx = 2 \cos \theta d\theta$, and

$$\begin{aligned} \int \frac{x^2}{\sqrt{4-x^2}} dx &= \int \frac{4(\sin \theta)^2}{2 \cos \theta} 2 \cos \theta d\theta \\ &= 4 \int (\sin \theta)^2 d\theta. \end{aligned}$$

Now the simplest thing to do is look in the table at the end of this exam to get integral as

$$4 \left(\frac{\theta}{2} - \frac{1}{2} \sin \theta \cos \theta \right) + C = 2\theta - 2 \sin \theta \cos \theta + C.$$

But now we need to substitute back to get an expression in terms of x . We have $\theta = \sin^{-1}(x/2)$. And also $\sin \theta = x/2$. But what is $\cos \theta$? We make a little triangle based on $\sin \theta = x/2$:



So the other side of the triangle is $\sqrt{4-x^2}$, and so $\cos \theta = \frac{\sqrt{4-x^2}}{2}$. Our final answer is then

$$2 \sin^{-1}(x/2) - 2 \frac{x\sqrt{4-x^2}}{2(2)} + C = 2 \sin^{-1}(x/2) - \frac{x\sqrt{4-x^2}}{2} + C$$

7. State whether the following series converge or diverge. Explain your answer. If you apply a test, you must give all details of the test to get full credit. (For example, for the comparison test, say what series you're comparing to.)

[4 points] (a) $\sum_{n=2}^{\infty} \frac{n^2 - 3}{n^4 + 3}$

Solution: The series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

is convergent by the p -test. Since

$$\frac{n^2 - 3}{n^4 + 3} \leq \frac{n^2}{n^4} = \frac{1}{n^2},$$

our series is convergent by the comparison test.

[4 points] (b) $\sum_{n=1}^{\infty} \frac{n-1}{n}$

Solution: Since $\frac{n-1}{n}$ converges to 1 as $n \rightarrow \infty$ and not to 0, this series diverges.

[8 points] 8. Compute $\int \frac{2x - 5}{(x - 2)^2(x - 3)} dx$.

Solution: We have to solve

$$\frac{2x - 5}{(x - 2)^2(x - 3)} = \frac{A}{x - 2} + \frac{B}{(x - 2)^2} + \frac{C}{x - 3}.$$

The first step is to clear fractions to get

$$2x - 5 = A(x - 2)(x - 3) + B(x - 3) + C(x - 2)^2.$$

Set $x = 3$ to get

$$1 = C.$$

Set $x = 2$ to get

$$-1 = -B,$$

which gives $B = 1$. Now set $x = 0$ (anything would work) to get

$$-5 = 6A - 3B + 4C = 6A - 3 + 4 = 6A + 1.$$

This gives $A = -1$.

Now, we integrate:

$$\begin{aligned} \int \frac{2x - 5}{(x - 2)^2(x - 3)} dx &= \int \left(-\frac{1}{x - 2} + \frac{1}{(x - 2)^2} + \frac{1}{x - 3} \right) dx \\ &= -\ln|x - 2| - \frac{1}{x - 2} + \ln|x - 3| + C. \end{aligned}$$

9. Find the following antiderivatives:

[3 points] (a) $\int \cos(x)\sqrt{\sin(x)} dx$

Solution: Set $u = \sin x$. Then $du = \cos x dx$. So,

$$\int \cos(x)\sqrt{\sin(x)} dx = \int \sqrt{u} du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\sin x)^{3/2}.$$

[3 points] (b) $\int \ln(x) dx$

Solution: Do integration by parts with $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x} dx$ and $v = x$, and

$$\begin{aligned} \int \ln x dx &= x \ln x - \int x \frac{1}{x} dx \\ &= x \ln x - x + C. \end{aligned}$$

TABLE OF TRIGONOMETRIC INTEGRALS

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C = \frac{x}{2} - \frac{1}{2} \sin x \cos x + C \quad \boxed{3}$$

$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C = \frac{x}{2} + \frac{1}{2} \sin x \cos x + C \quad \boxed{4}$$

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx \quad \boxed{5}$$

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx \quad \boxed{6}$$

$$\int \tan x \, dx = \ln |\sec x| + C = -\ln |\cos x| + C \quad \boxed{7}$$

$$\int \tan^m x \, dx = \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x \, dx \quad \boxed{8}$$

$$\int \cot x \, dx = -\ln |\csc x| + C = \ln |\sin x| + C \quad \boxed{9}$$

$$\int \cot^m x \, dx = -\frac{\cot^{m-1} x}{m-1} - \int \cot^{m-2} x \, dx \quad \boxed{10}$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C \quad \boxed{11}$$

$$\int \sec^m x \, dx = \frac{\tan x \sec^{m-2} x}{m-1} + \frac{m-2}{m-1} \int \sec^{m-2} x \, dx \quad \boxed{12}$$

$$\int \csc x \, dx = \ln |\csc x - \cot x| + C \quad \boxed{13}$$

$$\int \csc^m x \, dx = -\frac{\cot x \csc^{m-2} x}{m-1} + \frac{m-2}{m-1} \int \csc^{m-2} x \, dx \quad \boxed{14}$$

$$\int \sin mx \sin nx \, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + C \quad (m \neq \pm n) \quad \boxed{15}$$

$$\int \sin mx \cos nx \, dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} + C \quad (m \neq \pm n) \quad \boxed{16}$$

$$\int \cos mx \cos nx \, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + C \quad (m \neq \pm n) \quad \boxed{17}$$
