

*I pledge that I have neither given nor received  
unauthorized assistance during this examination.*

**Signature:**

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- You may use a scientific calculator on this exam, but you may not use a graphing calculator.
- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 6 problems on 8 pages.

| Question | Points | Score |
|----------|--------|-------|
| 1        | 12     |       |
| 2        | 8      |       |
| 3        | 12     |       |
| 4        | 12     |       |
| 5        | 16     |       |
| 6        | 12     |       |
| Total:   | 72     |       |

**Good luck!**

1. Find the derivatives of the following functions. You do **not** need to simplify your solutions.

[3 points]

(a)  $f(x) = x^2 \sin(2x)$

**Solution:**

$$f'(x) = 2x \sin(2x) + x^2 \cos(2x)(2)$$

[3 points]

(b)  $f(t) = \frac{2^t}{\ln t}$

**Solution:**

$$f'(t) = \frac{(\ln t)(2^t \log 2) - (2^t)\left(\frac{1}{t}\right)}{(\ln t)^2}$$

[3 points] (c)  $g(x) = \frac{1}{\sqrt[3]{x + \sqrt{x}}}$

**Solution:** On this one, it's best to view the function as  $g(x) = (x + \sqrt{x})^{-1/3}$ .

$$g'(x) = -\frac{1}{3}(x + \sqrt{x})^{-4/3}\left(1 + \frac{1}{2\sqrt{x}}\right)$$

[3 points] (d)  $y = \frac{x^{2x}(x-1)^3}{(3+5x)^4}$

**Solution:** This one is easiest using logarithmic differentiation. The first step is to compute

$$\log y = 2x \log x + 3 \log(x-1) - 4 \log(3+5x).$$

Now, we take derivatives of both sides:

$$\frac{y'}{y} = 2 \log x + 2x\left(\frac{1}{x}\right) + \frac{3}{x-1} - \frac{4}{3+5x}(5).$$

So, our final answer is

$$y' = \left(2 \log x + 2x\left(\frac{1}{x}\right) + \frac{3}{x-1} - \frac{4}{3+5x}(5)\right) \frac{x^{2x}(x-1)^3}{(3+5x)^4}$$

- [8 points] 2. Find the equation for the tangent line to the graph of  $f(x) = \sqrt{x^2 + 5}$  at the point  $(2, 3)$ .

**Solution:** First, we find the slope of the graph at  $x = 2$  by finding the derivative:

$$f'(x) = \frac{2x}{2\sqrt{x^2 + 5}}.$$

So,  $f'(2) = 2(2)/(2\sqrt{9}) = 2/3$ .

Using the point-slope form of a line, the tangent line at  $(2, 3)$  is

$$y - 3 = \frac{2}{3}(x - 2).$$

- [12 points] 3. A spherical snowball melts so that its surface area decreases at a rate of  $1 \text{ cm}^2/\text{min}$ . (The surface area of a sphere with radius  $r$  is  $4\pi r^2$ .) What is the rate of change of the radius of the snowball when it is equal to 5 cm?

**Solution:** Let  $r$  be the radius of the ball and let  $S$  be the surface area.

**Know:**  $\frac{dS}{dt} = -1 \text{ cm}^2/\text{min}$

**Want:**  $\frac{dr}{dt}$  when  $r = 5$

**Relationship:**  $S = 4\pi r^2$

Differentiate both sides of the relationship with respect to  $t$ :

$$\frac{dS}{dt} = 4\pi(2r)\frac{dr}{dt} = 8\pi r \frac{dr}{dt}.$$

Plug in known data:

$$-1 = 8\pi(5)\frac{dr}{dt}.$$

Solving for  $\frac{dr}{dt}$ , we get

$$\frac{dr}{dt} = -\frac{1}{40\pi}.$$

4. Consider the graph of  $y = 10x^3 - x^5$ .

[4 points]

(a) List the intervals on which this function is increasing.

**Solution:** The derivative is

$$y' = 30x^2 - 5x^4 = 5x^2(6 - x^2).$$

So, the critical points are  $x = \pm\sqrt{6}$  and  $x = 0$ . We then check that  $y'$  is negative when  $x < -\sqrt{6}$  or  $x > \sqrt{6}$ , and  $y'$  is positive when  $-\sqrt{6} < x < 0$  or  $0 < x < \sqrt{6}$ . So,  $y$  is increasing on the intervals  $(-\sqrt{6}, 0)$  and  $(0, \sqrt{6})$ . (It's decreasing on  $(-\infty, \sqrt{6})$  and  $(\sqrt{6}, \infty)$ .)

[1 point]

(b) Give the  $x$ -coordinates of all local minima, or say that there are none.

[1 point]

(c) Give the  $x$ -coordinates of all local maxima, or say that there are none.

**Solution:** By the first derivative test, there is a local minimum at  $x = -\sqrt{6}$  and a local maximum at  $x = \sqrt{6}$ .

- [4 points] (d) List the intervals on which this function is concave up.

**Solution:** First, we find the second derivative:

$$y'' = 60x - 20x^3 = 20x(3 - x^2).$$

So,  $y'' = 0$  at  $x = \pm\sqrt{3}$  and at  $x = 0$ . We then figure out that  $y''$  is positive on  $(-\infty, -\sqrt{3})$ , negative on  $(-\sqrt{3}, 0)$ , positive on  $(0, \sqrt{3})$ , and negative on  $(\sqrt{3}, \infty)$ . So,  $y$  is concave up on  $(-\infty, -\sqrt{3})$  and on  $(0, \sqrt{3})$ .

- [2 points] (e) List the  $x$ -coordinates of all inflection points.

**Solution:** There are inflection points where  $y''$  changes signs. From the previous question, this happens at  $x = \pm\sqrt{3}$  and at  $x = 0$ .

- [8 points] 5. (a) Find the slope of the tangent line to the curve defined by

$$x^2y + y^4 = 4 + 2x$$

at the point  $(-1, 1)$ .

**Solution:** Using implicit differentiation,

$$2xy + x^2 \frac{dy}{dx} + 4y^3 \frac{dy}{dx} = 2.$$

Solving for  $\frac{dy}{dx}$ , we get

$$\frac{dy}{dx} = \frac{2 - 2xy}{x^2 + 4y^3}.$$

At  $(-1, 1)$ , this is  $4/5$ .

- [8 points] (b) Using linear approximation, estimate a  $y$ -value on this curve when  $x = -0.9$ .

**Solution:** We know that the curve contains the point  $(-1, 1)$ , at which it has slope  $4/5$ . The change from  $x = -1$  to  $x = -0.9$  is  $0.1$ . So, the linear approximation for the  $y$ -value of the curve at  $x = -0.9$  is

$$1 + (0.1)\frac{4}{5} = 1.08.$$

- [12 points] 6. Find the minimum and maximum value taken by the function  $f(x) = 2\sqrt{x} - x$  in the interval  $[0, 3]$ .

**Solution:** First, we find all critical points in the interval:

$$f'(x) = \frac{1}{\sqrt{x}} - 1 = 0,$$

which has solution  $x = 1$ . So, the maximum and minimum on  $[0, 3]$  must occur either at  $x = 1$  or at one of the endpoints  $x = 0$  and  $x = 3$ . Evaluating, we find

$$f(0) = 0$$

$$f(1) = 1$$

$$f(3) = 2\sqrt{3} - 3 \approx 0.464.$$

So, the minimum of the function is 0 and the maximum is 1.