Math 232, Exam 1 Name: ______________________

September 20, 2023

> *I pledge that I have neither given nor received unauthorized assistance during this examination.*
>
> **Signature:**

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- You may use a scientific calculator, but not a graphing calculator or phone.
- It is okay to leave a numerical answer like $\frac{39}{2} - (18 - e^2)$ unsimplified.
- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 7 problems on 6 pages.

Question	Points	Score
1	9	
2	8	
3	6	
4	6	
5	6	
6	8	
7	8	
Total:	51	

Good luck!

[9 points] 1. Find the following antiderivatives:

(a) $\displaystyle\int \frac{x}{\sqrt{4x^2+3}}\,dx$

Solution: Substitute $u = 4x^2+3$. Then $du = 8x\,dx$, and

$$\int \frac{x}{\sqrt{4x^2+3}}\,dx = \frac{1}{8}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{8}\int u^{-1/2}\,du = \frac{2u^{1/2}}{8} + C = \frac{\sqrt{4x^2+3}}{4} + C.$$

(b) $\displaystyle\int \frac{3x^2+6x+1}{x}\,dx$

Solution:

$$\int \frac{3x^2+6x+1}{x}\,dx = \int \left(3x+6+\frac{1}{x}\right)du = \frac{3x^2}{2} + 6x + \ln|x| + C$$

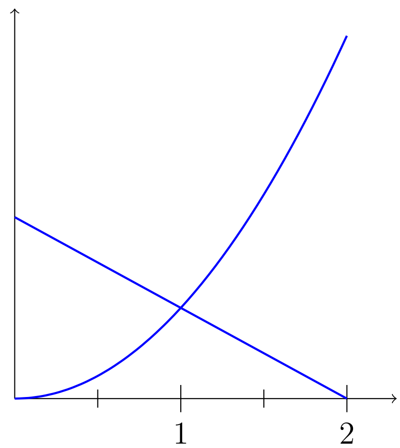
(c) $\displaystyle\int e^{2x}\sqrt{e^{2x}+2}\,dx$

Solution: Let $u = e^{2x}+2$. Then $du = 2e^{2x}\,dx$, and

$$\int e^{2x}\sqrt{e^{2x}+2}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}\left(e^{2x}+2\right)^{3/2} + C.$$

[8 points] 2. (a) On the axes below, sketch the curves $y = x^2$ and $y = 2 - x$ on the interval $[0, 2]$.

Solution:



(b) Compute the area of the region bounded by the curves $y = x^2$, $y = 2 - x$, and the x-axis.

Solution: The bottom of the region is always the x-axis. The top of the region is the curve $y = x^2$ from 0 to 1 and the curve $y = 2 - x$ from 1 to 2. So its area is

$$\int_0^1 x^2\,dx + \int_1^2 (2 - x)\,dx.$$

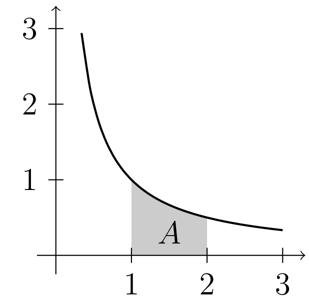
The first integral is

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\bigg|_0^1 = \frac{1}{3}.$$

The second integral gives the area of a triangle with base 1 and height 1 and is hence 1/2. So the total area is

$$\frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

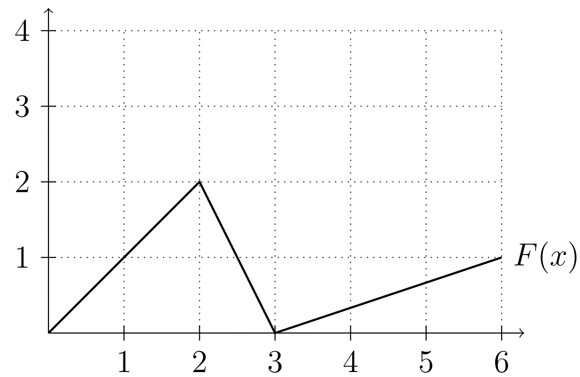
[6 points] 3. Let A be the region underneath the function $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 2$. Find the volume of the solid obtained by rotating A around the x-axis.



Solution: The cross section at x is a circle with radius $1/x$. Integrating the areas of these circles from $x = 1$ to $x = 2$ gives

$$\int_1^2 \pi \left(\frac{1}{x}\right)^2 dx = \pi \int_1^2 x^{-2}\,dx = -\pi x^{-1}\Big|_1^2 = -\pi\left(\frac{1}{2} - 1\right) = \frac{\pi}{2}$$

[6 points] 4. Let $F(x) = \int_0^x f(t)\,dt$. Here is a picture of the graph of $F(x)$:



Be careful! This is a picture of $F(x)$, not a picture of $f(x)$.

Are the following statements true or false?

(a) $f(1) > 0$

√ **True** ◯ False ◯ Not enough info

(b) $F(1) > 0$

√ **True** ◯ False ◯ Not enough info

(c) $f(4) > 0$

√ **True** ◯ False ◯ Not enough info

(d) $f(x) \geq 0$ for all choices of x

◯ True √ **False** ◯ Not enough info

(e) The maximum value of $f(x)$ on the interval $[0, 6]$ is 2.

◯ True √ **False** ◯ Not enough info

(f) The minimum value of $f(x)$ on the interval $[0, 6]$ is -2.

√ **True** ◯ False ◯ Not enough info

Solution: The key thing in this problem is to realize that $f(x)$ is the derivative of $F(x)$. So, for example, the maximum value of $f(x)$ on the interval is 1 (the largest slope of $F(x)$) and the minimum value of $f(x)$ on the interval is -2 (the smallest slope of $F(x)$).

[6 points] 5. What is the average value of the function $\sin x$ on the interval from 0 to π?

Solution: The average value is

$$\frac{1}{\pi}\int_0^{\pi} \sin x\,dx = -\frac{1}{\pi}\cos x\Big|_0^{\pi} = -\frac{1}{\pi}(\cos\pi - \cos 0) = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}.$$

[8 points] 6. Using integration, find the volume of a cone with height 2 whose base is a circle with radius 2.

Solution: By similar triangles, the cross section at distance h from the tip of the cone is a circle with radius h. Adding up all the areas of these circles with an integral gives a volume of

$$\int_0^2 \pi h^2\,dh = \frac{\pi h^3}{3}\Big|_0^2 = \frac{8\pi}{3}.$$

[8 points] 7. Find the following definite integrals:

(a) $\int_{-4}^{4} |x|\, dx$

Solution: It's easiest to do this geometrically. The area under the curve is made up of two triangles, each with base and height 4. So the area is $8 + 8 = 16$.

(b) $\int_{4}^{5} (2t-8)^4\, dt$

Solution: Substitute $u = 2t - 8$. Then $du = 2\, dt$, and

$$\int_{4}^{5} (2t-8)^4\, dt = \frac{1}{2} \int_{2(4)-8}^{2(5)-8} u^4\, du = \left. \frac{u^5}{2(5)} \right|_{0}^{2} = \frac{16}{5}$$