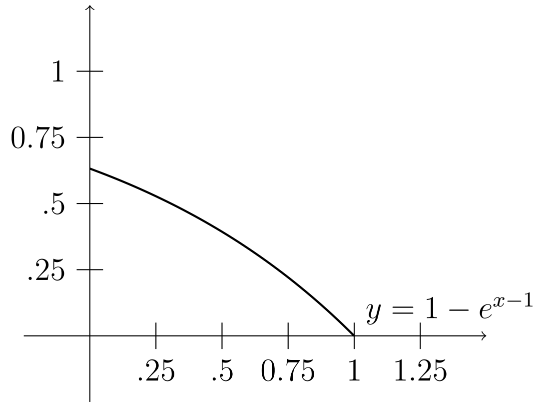
1. Here is a graph of the curve $y = 1 - e^{x-1}$ from $x = 0$ to $x = 1$:



(a) Find the area of the region under the curve.

Solution: Just compute

$$\int_0^1 (1 - e^{x-1})\,dx = (x - e^{x-1})\Big|_0^1 = (1 - e^0) - (0 - e^{-1}) = e^{-1}.$$

Now, take this region and rotate it around the y-axis to sweep out a solid of revolution.

(b) Find the volume of this solid using the disc/washer method.

Solution: The region to be rotated extends from $x = 0$ to $x = 1$ and from $y = 0$ to $y = 1 - e^{-1}$.

When we rotate around the y-axis, using discs we integrate with respect to y, so from 0 $1 - e^{-1}$. We'll need to rewrite the curve with x as a function of y:

$$x = \log(1 - y) + 1.$$

The radius of the disc at height y is the distance from the curve to the y-axis, which is $1 + \log(1 - y)$. So, the integral is

$$\int_0^{1-e^{-1}} \pi\big(1 + \ln(1 - y)\big)^2\,dy.$$

Let's substitute $x = 1 - y$, using $dx = -dy$. This won't really help, but it will make the integral slightly neater. This gets us to

$$-\int_1^{e^{-1}} \pi(1 + \ln x)^2\,du = \int_{e^{-1}}^1 \pi\left(1 + 2\ln x + (\ln x)^2\right)dx.$$

You might just know the antiderivative of $\ln x$, but we can get it by integration by parts with $u = \ln x$ and $dv = dx$:

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} \, dx = x \ln x - x + C.$$

To find the antiderivative of $(\ln x)^2$, we use integration by parts with $u = (\ln x)^2$ and $dv = dx$. We have $du = 2(\ln x)\frac{1}{x} \, dx$ and $v = x$, and

$$\begin{aligned} \int (\ln x)^2 \, dx = uv - \int v \, du &= x(\ln x)^2 - \int 2(\ln x)\frac{1}{x} x \, dx \\ &= x(\ln x)^2 - 2\int \ln x \, dx \\ &= x(\ln x)^2 - 2x \ln x + 2x + C. \end{aligned}$$

Putting it all together,

$$\begin{aligned} \int_0^{1-e^{-1}} \pi\big(1 + \ln(1-y)\big)^2 \, dy &= \int_{e^{-1}}^1 \pi\Big(1 + 2\ln x + (\ln x)^2\Big) \, dx \\ &= \pi\Big(x + 2x\ln x - 2x + x(\ln x)^2 - 2x\ln x + 2x\Big)\Big|_{e^{-1}}^1 \\ &= \pi\Big(x + x(\ln x)^2\Big)\Big|_{e^{-1}}^1 \\ &= \pi\big(1 + 0 - (e^{-1} + e^{-1}(-1)^2)\big) = \pi(1 - 2e^{-1}). \end{aligned}$$

(c) Find the volume of this solid using the cylindrical shells method.

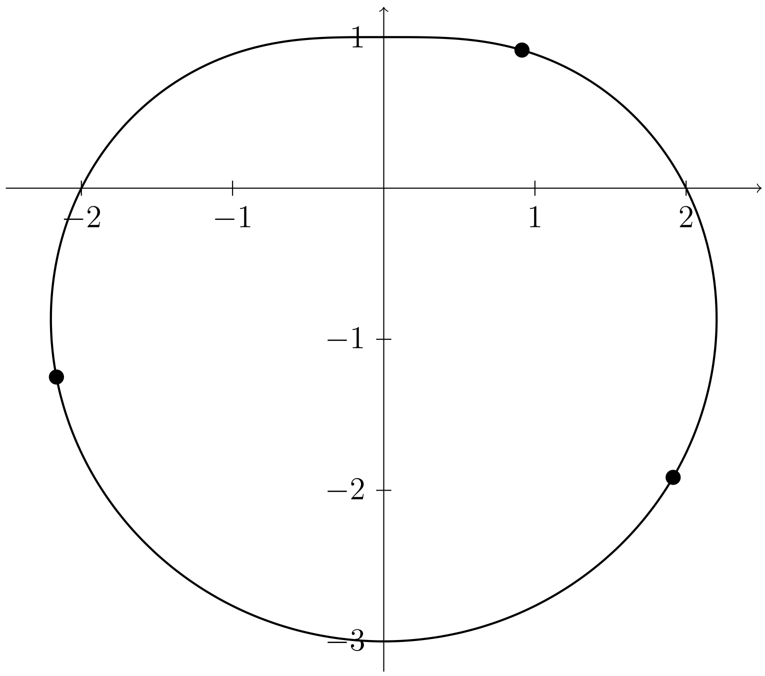
Solution: With shells and a solid of revolution around the y-axis, we integrate with respect to x. So we go from $x = 0$ to $x = 1$. The radius of the shell at x is x, and its height is $1 - e^{x-1}$. Now the integral is

$$\int_0^1 2\pi x(1 - e^{x-1}) \, dx.$$

Do integration by parts with $u = x$, $dv = (1 - e^{x-1}) \, dx$. Then $du = dx$ and $v = x - e^{x-1}$, and the integral becomes

$$\begin{aligned} 2\pi\left(x(x - e^{x-1})\Big|_0^1 - \int_0^1 (x - e^{x-1}) \, dx\right) &= 2\pi\left(0 - \left(\frac{x^2}{2} - e^{x-1}\right)\Big|_0^1\right) \\ &= -2\pi\left(\left(\frac{1}{2} - 1\right) - \left(0 - e^{-1}\right)\right) \\ &= \pi(1 - 2e^{-1}). \end{aligned}$$

2. Here is a graph of the polar equation $r = 2 - \sin\theta$. The points on the graph are plotted at locations $\theta = \pi/4$, $\theta = 7\pi/6$, and $\theta = 7\pi/4$.



(a) What are the polar coordinates of the three points?

Solution: We're given the values of θ, and we just plug them into the given equation to get the values of r:

$$(r, \theta) = \left(2 - \frac{\sqrt{2}}{2}, \frac{\pi}{4}\right),$$
$$(r, \theta) = \left(\frac{5}{2}, \frac{7\pi}{6}\right),$$

and

$$(r, \theta) = \left(2 + \frac{\sqrt{2}}{2}, \frac{7\pi}{4}\right),$$

(b) What are the rectangular coordinates of the three points? (Give exact answers, not decimal approximations; you'll need to know the special values of $\sin\theta$ and $\cos\theta$ that can be computed exactly as square roots.)

> **Solution:** They are
>
> $$\left(\left(2-\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2},\ \left(2-\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}\right)$$
> $$\left(-\frac{5}{2}\left(\frac{\sqrt{3}}{2}\right),\ -\frac{5}{2}\left(\frac{1}{2}\right)\right)$$
>
> and
>
> $$\left(\left(2+\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2},\ -\left(2+\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}\right)$$

(c) Find the area enclosed by the curve.

> **Solution:** We use the formula
>
> $$\begin{aligned}\frac{1}{2}\int_0^{2\pi} r^2\,d\theta &= \frac{1}{2}\int_0^{2\pi}(2-\sin\theta)^2\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\left(4-4\sin\theta+\sin^2\theta\right)d\theta \\ &= \frac{1}{2}\left(8\pi+4\cos\theta\Big|_0^{2\pi}+\left(\frac{\theta}{2}-\frac{\sin(2\theta)}{4}\right)\Big|_0^{2\pi}\right) \\ &= \frac{1}{2}(8\pi+\pi)=\frac{9\pi}{2}.\end{aligned}$$

3. Do the following series converge? Explain your answer. If you apply a test, you must give all details of the test to get full credit. (For example, for the comparison test, say what series you're comparing to. For the ratio or root test, give the value of the limit you compute when applying the test.)

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{4^n}$

> **Solution:** The ratio of successive terms is
>
> $$\frac{(n+1)^3}{4^{n+1}}\frac{4^n}{n^3}=\frac{1}{4}\left(\frac{n+1}{n}\right)^3=\frac{1}{4}\left(1+\frac{1}{n}\right)^3,$$
>
> which has limit $1/4$ as $n\to\infty$. Since this limit is less than 1, the series converges by the ratio test.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{1+2^n+3^n}{5^n}$

Solution: The ratio of successive terms is

$$\frac{1+2^{n+1}+3^{n+1}}{5^{n+1}}\frac{5^n}{1+2^n+3^n} = \left(\frac{1+2^{n+1}+3^{n+1}}{1+2^n+3^n}\right)\frac{1}{5}$$
$$= \frac{3^{-n}+2(\frac{2}{3})^n+3}{3^{-n}+(\frac{2}{3})^n+1}\left(\frac{1}{5}\right).$$

The limit of this as $n \to \infty$ is $3/5$ which is less than 1. So the sum converges by the ratio test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2^n+1}}$

Solution: The series is alternating, and the nonnegative part $1/\sqrt{2^n+1}$ is decreasing and tends to 0 as $n \to \infty$. So the series converges by the alternating series test.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+3}{3n-5}$

Solution: The limit of the nth term of the series is $2/3$. Since this is not equal to 0, the series diverges.

4. Consider the curves $y = x$ and $y = xe^{5x}$. Find the area between the two curves from $x = 0$ to $x = 2$.

Solution: From $x = 0$ to $x = 2$, the curve $y = xe^{5x}$ is always above $y = x$. To see this, we could solve for the intersection points of the two curves by setting $x = xe^{5x}$, then rewriting this equation as $x(e^{5x} - 1) = 0$, whose only solution is $x = 0$. So the curves intersect at $x = 0$, but at $x = 2$ the line $y = x$ is at y-coordinate 2, while the curve $y = xe^{5x}$ is at y-coordinate $2e^{10} \approx 44052.9$, which is much bigger. So the curve $y = xe^{5x}$ is always on top.

So, the area between the two curves is

$$\int_0^2 \left(xe^{5x} - x\right) dx = \int_0^2 xe^{5x}\,dx - \frac{x^2}{2}\bigg|_0^2 = \int_0^2 xe^{5x}\,dx - 2.$$

Now we use integration by parts to integrate xe^{5x}, setting $u = x$ and $dv = e^{5x}\,dx$.

Then we have $du = dx$ and $v = \frac{1}{5}e^{5x}$, and we get

$$\begin{aligned}\int_0^2 xe^{5x}\,dx &= uv\Big|_0^2 - \int_0^2 v\,du \\ &= \frac{1}{5}xe^{5x}\Big|_0^2 - \int_0^2 \frac{1}{5}e^{5x}\,dx \\ &= \frac{2}{5}e^{10} - \frac{1}{25}e^{5x}\Big|_0^2 \\ &= \frac{2}{5}e^{10} - \frac{1}{25}(e^{10}-1) = \frac{9}{25}e^{10} - \frac{1}{25},\end{aligned}$$

and

$$\int_0^2 \left(xe^{5x} - x\right)dx = \frac{9}{25}e^{10} - \frac{1}{25} - 2 \approx 7927.57.$$

5. A particle moves on a plane and is at location $\big(x(t), y(t)\big)$ where

$$\begin{aligned}x(t) &= -t, \\ y(t) &= t^2 - 5t + 2\end{aligned}$$

(a) Is the particle ever at position $(-1,-2)$? If so, at what time?

Solution: The question is whether there's a time t so that $x(t) = -1$ and $y(t) = -2$. The only solution to $x(t) = -1$ is $t = 1$, so that's our only candidate. We then check whether $y(1) = -2$. Since it is, the particle is at position $(-1,-2)$ at time $t = 1$.

(b) What is the particle's speed at time $t = -3$?

Solution: First we find

$$\begin{aligned}x'(t) &= -1, \\ y'(t) &= 2t - 5.\end{aligned}$$

Now the speed at time $t = -3$ is given by

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{(-1)^2 + (-11)^2} = \sqrt{122}.$$

(c) Set up but do not compute an integral that determines how far the particle moves between times 0 and 1. (If you want a challenge, also compute the integral.)

Solution: The integral is

$$\int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^1 \sqrt{1 + (2t-5)^2}\,dt$$

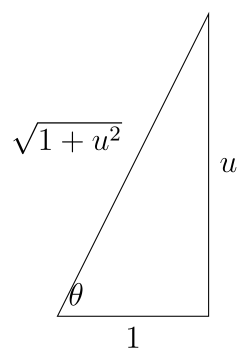
If you want to do this integral, first substitute $u = 2t - 5$:

$$\int_0^1 \sqrt{1 + (2t-5)^2}\,dt = \frac{1}{2}\int_{-5}^{-3} \sqrt{1+u^2}\,du.$$

Now make the substitution $u = \tan\theta$ to find

$$\begin{aligned}\int \sqrt{1+u^2}\,du &= \int \sqrt{1+\tan^2\theta}\sec^2\theta\,d\theta \\ &= \int \sec^3\theta\,d\theta = \frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.\end{aligned}$$

Now we have to substitute to replace θ by u. We have $\tan\theta = u$ and we make a triangle:



So $\sec\theta = \sqrt{1+u^2}$. Thus

$$\int \sqrt{1+u^2}\,du = \frac{u\sqrt{1+u^2}}{2} + \frac{1}{2}\ln|\sqrt{1+u^2} + u| + C$$

Finally,

$$\begin{aligned}\int_0^1 \sqrt{1+(2t-5)^2}\,dt &= \frac{1}{2}\int_{-5}^{-3} \sqrt{1+u^2}\,du \\ &= \frac{1}{2}\left(\frac{u\sqrt{1+u^2}}{2} + \frac{1}{2}\ln|\sqrt{1+u^2}+u|\right)\Bigg|_{-5}^{-3} \approx 4.126.\end{aligned}$$

6. Say if each statement is true or false:
   (a) If $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\,dx$ converges.

Solution: False

(b) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Solution: True

(c) The technique of partial fractions can be used to find $\int \frac{1}{(x-2)(x-3)}\,dx$.

Solution: True

(d) The interval $[2,4]$ might be the interval of convergence for a power series

$$\sum_{n=0}^{\infty} a_n(x-3)^n.$$

Solution: True

(e) The interval $[2,5)$ might be the interval of convergence for a power series

$$\sum_{n=0}^{\infty} a_n(x-3)^n.$$

Solution: False

(f) The interval $(-\infty,\infty)$ might be the interval of convergence for a power series

$$\sum_{n=0}^{\infty} a_n(x-3)^n.$$

Solution: True

(g) The interval $(1,5]$ might be the interval of convergence for a power series

$$\sum_{n=0}^{\infty} a_n(x-3)^n.$$

Solution: True

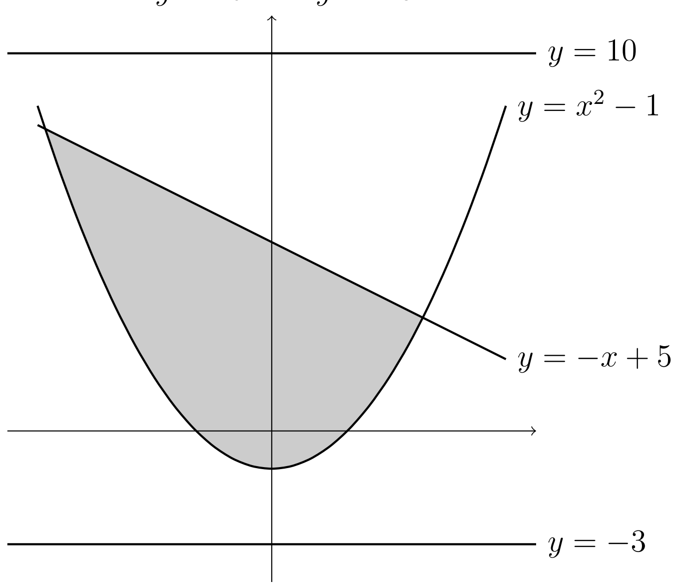
(h) The technique of trigonometric substitution can be used to find $\int e^{-x^2}\,dx$.

Solution: False

7. Set up **but do not solve** integrals to find the volume of the solid obtained by revolving the region bounded between the curves $y = x^2 - 1$ and the line $y = -x + 5$ about the following lines:

(a) $y = 10$ (use disc/washer method)

Solution: A sketch would be helpful. We start by finding the intersections of $y = x^2 - 1$ and $y = -x + 5$ by setting $x^2 - 1 = -x + 5$ and solving, which yields a quadratic equation with solutions $x = -3$ and $x = 2$. Here's a sketch, along with the axes of rotation $y = 10$ and $y = -3$:



For the disc/washer method, we will have washers whose outer radius is the distance from the bottom of the region up to the line $y = 10$ and whose inner radius is the distance from the top of the region up to the line $y = 10$. Here's the integral:

$$\int_{-3}^{2} \pi\Big((x^2 - 1 - 10)^2 - (-x + 5 - 10)^2\Big)\,dx$$

(b) $y = -3$ (use shell method)

Solution: For the cylindrical shells method, we'll be integrating with respect to y from the bottom of the region to the top. The bottom is at $y = -1$, the value of the parabola at $x = 0$. The top is at $y = 8$, the y-value of the two curves at $x = -3$. For the shell at some y-value, the height of the shell is the distance from the right to the left of the region at that y-coordinate, and the radius of the shell is the distance from y to -3 (since we're rotating around the line $y = -3$. We need to break the integral into two pieces, one from $y = -1$ to $y = 3$, the y-coordinate of the right intersection point of the two curves, since the right boundary of the region switches at $y = 3$ from the parabola to the line. And we need to rewrite the curves with x in terms of y:

$$x = \pm\sqrt{y+1},$$
$$x = 5 - y$$

Here's the integral:

$$\int_{-1}^{3} 2\pi(y+3)\big(\sqrt{y+1} - (-\sqrt{y+1})\big)\,dy + \int_{3}^{8} 2\pi(y+3)\big(5 - y - (-\sqrt{y+1})\big)\,dy$$
$$= \int_{-1}^{3} 2\pi(y+3)\big(2\sqrt{y+1}\big)\,dy + \int_{3}^{8} 2\pi(y+3)\big(5 - y + \sqrt{y+1}\big)\,dy.$$

8. Compute the following integrals:

(a) $\displaystyle\int_1^{e^2} \frac{4+\ln x}{x}\,dx$

Solution: Substitute $u = 4 + \ln x$ to get

$$\int_1^{e^2} \frac{4+\ln x}{x}\,dx = \int_4^6 u\,du = \frac{u^2}{2}\bigg|_4^6 = \frac{36-16}{2} = 10.$$

(b) $\displaystyle\int \frac{e^t}{e^{2t}+3e^t+2}\,dt$

Solution: Substitute $u = e^t$ to get

$$\int \frac{e^t}{e^{2t}+3e^t+2}\,dt = \int \frac{1}{u^2+3u+2}\,du.$$

Now do partial fractions, rewriting

$$\frac{1}{u^2+3u+2} = \frac{A}{x+2} + \frac{B}{x+1},$$

clearing fractions and substituting $u = -2$ and $u = -1$ to get $A = -1$ and $B = 1$, which gives

$$\int \frac{1}{u^2 + 3u + 2}\,du = -\ln|x+2| + \ln|x+1| + C.$$

Substituting back,

$$\int \frac{e^t}{e^{2t} + 3e^t + 2}\,dt = -\ln|e^t+2| + \ln|e^t+1| + C.$$

Note that you can't simplify these logarithms in any useful way.

(c) $\displaystyle\int_1^\infty \frac{1}{(2x+1)^3}\,dx$

Solution:

$$\begin{aligned}\int_1^\infty \frac{1}{(2x+1)^3}\,dx &= \lim_{R\to\infty} -\frac{1}{4(2x+1)^2}\bigg|_1^R \\ &= \lim_{R\to\infty}\left(-\frac{1}{4(2R+1)^2} + \frac{1}{36}\right) = \frac{1}{36}.\end{aligned}$$

(d) $\displaystyle\int \frac{9t+1}{3t+4}\,dt$

Solution:

$$\begin{aligned}\int \frac{9t+1}{3t+4}\,dt &= \int \left(3 - \frac{11}{3t+4}\right)dt \\ &= 3t - \frac{11}{3}\ln|t + 4/3| + C.\end{aligned}$$