Math 311, Version A, Final
December 17, 2025

Name: ____________________________

I pledge that I have neither given nor received unauthorized assistance during this examination.

Signature:

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- No calculators or electronic devices are allowed.
- Unless the problem specifically says, you **do not** need to calculate the final answer. An answer that looks something like

  $$\frac{\binom{6}{2}\binom{8}{5}}{\binom{23}{4}} \quad \text{or} \quad \frac{(12)(11)(10)+(11)(10)(9)}{14!}$$

  is entirely acceptable.
- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 9 problems on 11 pages.

Question	Points	Score
1	16	
2	9	
3	6	
4	14	
5	12	
6	6	
7	16	
8	1	
9	7	
Total:	87	

Good luck!

1. A factory makes bolts designed to attach pieces of metal under high pressure. They model their production by assuming that each bolt is defective with probability 1/75, independent of all other bolts.

[2 points] (a) Two pieces of metal are attached with 20 bolts. What is the expected number of defective bolts out of these 20 bolts?

Solution: This is the expectation of the $\mathrm{Bin}(20, 1/75)$ distribution, or 20/75.

[4 points] (b) The two pieces will separate under pressure if 2 or more out of these 20 bolts are defective. What is the probability of this?

Solution: The simplest way to write this is

$$1 - \left(\frac{74}{75}\right)^{20} - 20\left(\frac{74}{75}\right)^{19}\left(\frac{1}{75}\right)$$

[4 points] (c) Suppose that the two pieces do not separate under pressure, i.e., that at most 1 of the 20 bolts is defective. Given that this occurs, what is the probability that none of the 20 bolts are defective?

Solution: This conditional probability is

$$\frac{\left(\frac{74}{75}\right)^{20}}{\left(\frac{74}{75}\right)^{20} + 20\left(\frac{74}{75}\right)^{19}\left(\frac{1}{75}\right)} = \frac{37}{47}$$

This problem continues onto the next page.

[3 points] (d) The factory opens in the morning and starts producing bolts. Let X be the number of bolts produced *before* the first defective one. What is $E(X)$? (Note: do not include the defective bolt in your count X. That is, if the first bolt is good and the second is defective, then X is 1 rather than 2.

Solution: This is the expectation of the version of the geometric distribution that starts at 0, with parameter 1/75. This gives it expectation

$$\frac{1}{1/75} - 1 = 74.$$

[3 points] (e) For the random variable X from the previous question, find $P(X = 10)$. That is, what is the probability that exactly 10 good bolts are produced before the first defective one?

Solution: It's

$$\left(\frac{74}{75}\right)^{10}\left(\frac{1}{75}\right).$$

[9 points] 2. Let X be the number of hours past noon at which a fire drill takes place. The drill will happen sometime between 1pm and 4pm, but not from 2pm to 3pm, and it is equally likely to occur at any of the permitted times. (That is, X is uniformly distributed on the set $[1,2] \cup [3,4]$.)

(a) What is the probability density function of X?

Solution: It needs to be constant on $[1,2] \cup [3,4]$ and 0 outside of there. To integrate to 1, it then has to be

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [1,2] \cup [3,4], \\ 0 & \text{otherwise.} \end{cases}$$

(b) What is the expectation of X?

Solution: By symmetry, $E(X) = 2.5$. Or compute it as

$$E(X) = \int_1^2 \tfrac{1}{2}x\,dx + \int_3^4 \tfrac{1}{2}x\,dx = \tfrac{1}{4}(2^2 - 1^2) + \tfrac{1}{4}(4^2 - 3^2) = \tfrac{3}{4} + \tfrac{7}{4} = \tfrac{5}{2}.$$

(c) What is the variance of X?

Solution: First we compute

$$E(X^2) = \int_1^2 \tfrac{1}{2}x^2\,dx + \int_3^4 \tfrac{1}{2}x^2\,dx = \tfrac{1}{6}(2^3 - 1^3) + \tfrac{1}{6}(4^3 - 3^3) = \tfrac{7}{6} + \tfrac{37}{6} = \tfrac{44}{6}.$$

Now

$$\operatorname{Var}(X) = E(X^2) - E(X)^2 = \tfrac{44}{6} - \tfrac{25}{4} = \tfrac{13}{12}.$$

[6 points] 3. Suppose that $X_1, \ldots, X_{200}$ are i.i.d. with mean 1 and variance 2. Use the normal approximation to estimate $P(X_1 + \cdots + X_{200} > 150)$. **Give your answer in terms of $\Phi(x)$, the cumulative distribution function of the standard normal distribution.**

Solution: Let $S \sim X_1 + \cdots + X_{200}$. Then $\mu = E(S) = 200(1) = 200$ and $\sigma^2 = \mathrm{Var}(S) = 200(2) = 400$. Since S is a sum of i.i.d. random variables, the CLT applies and states that we can approximate $P(S > 150)$ as

$$\begin{aligned} P(S > 150) &= P\left(\frac{S-\mu}{\sigma} > \frac{150-\mu}{\sigma}\right) \\ &= P\left(\frac{S-200}{20} > \frac{-50}{20}\right) \\ &\approx P\left(Z > \frac{-5}{2}\right) = 1 - \Phi(-2.5), \end{aligned}$$

where Z has the standard normal distribution.

4. A room has bins labeled $1, \ldots, 20$. Each of 20 balls is placed into a bin chosen uniformly at random, independently of each other.

[2 points] (a) Let X_1 be the number of balls placed into bin #1. What is the **exact** distribution of X_1? (Your answer should be a named distribution and you should give its parameter(s).)

Solution: $\mathrm{Bin}(20, 1/20)$

[2 points] (b) What distribution is appropriate to **approximate** the distribution of X_1? (Your answer should be a named distribution and you should give its parameter(s).)

Solution: $\mathrm{Poi}(1)$

[3 points] (c) What is the **exact** probability that bin #1 is empty?

Solution: You could think about this in terms of the binomial distribution, but you don't really have to. Each ball is placed outside of bin #1 with probability 19/20. So the probability that all 20 balls are placed outside of bin #1 is

$$\left(\frac{19}{20}\right)^{20} \approx .3585.$$

Note that this is indeed close to the probability placed on 0 by the $\mathrm{Poi}(1)$ distribution, which is

$$e^{-1} \approx .368.$$

This problem continues onto the next page.

[3 points] (d) What is the expected number of empty bins?

Solution: Let J_i be 1 if bin $\#i$ is empty and otherwise 0. Then

$$E(J_i) = P(\text{bin \#i is empty}) = \left(\frac{19}{20}\right)^{20}$$

from the previous problem. The total expected number of empty bins is

$$E(J_1 + \cdots + J_{20}) = E(J_1) + \cdots + E(J_{20}) = 20\left(\frac{19}{20}\right)^{20} \approx 7.170.$$

[4 points] (e) What is the probability that every bin contains exactly one ball?

Solution: There's an approach using the techniques from chapter 1 or using conditional probability in the style of the birthday problem. We'll use the latter approach.

Let E_i be the event that the first i balls are all placed into different bins. Our goal is to compute E_{20}. Observe that

$$P(E_{i+1} \mid E_i) = \frac{20 - i}{20},$$

since if i balls are placed all in different bins, then there are $20 - i$ free bins left to place the next ball without putting two in the same bin.

Now, the probability of E_{20} is

$$\begin{aligned} P(E_{20}) &= P(E_1)P(E_2 \mid E_1) \cdots P(E_{20}) \\ &= \frac{20}{20} \cdot \frac{19}{20} \cdot \cdots \cdot \frac{1}{20} = \frac{20!}{20^{20}} = .000000023201961 \end{aligned}$$

[12 points] 5. Suppose that X and Y have joint probability density function

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the marginal probability density function of X. **Fully compute all integrals in this answer.**

Solution:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{3}{2}\left(x^2+y^2\right)dy$$
$$= \frac{3}{2}\left(x^2 y + \frac{y^3}{3}\right)\Big|_0^1 = \frac{3}{2}\left(x^2 + \frac{1}{3}\right) = \frac{3}{2}x^2 + \frac{1}{2},$$

for $0 \le x \le 1$.

(b) Are X and Y independent? Justify your answer.

Solution: No, they're not. We have $f_X(x) = \frac{3}{2}x^2 + \frac{1}{2}$ on $[0,1]$, and by symmetry we also have $f_Y(y) = \frac{3}{2}y^2 + \frac{1}{2}$ on $[0,1]$. But these two marginal pdfs multiply to

$$\tfrac{9}{4}x^2y^2 + \tfrac{3}{4}x^2 + \tfrac{3}{4}y^2 + \tfrac{1}{4},$$

for $0 \le x, y \le 1$, which is different from $f(x,y)$.

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(c) Set up an integral to find $P(Y > X)$. **Your answer should be a parametrized, double integral, but don't compute it.**

Solution:

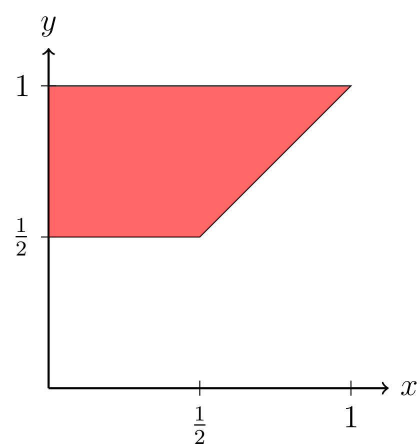
$$\int_0^1 \int_x^1 \frac{3}{2}(x^2 + y^2)\, dy\, dx$$

or

$$\int_0^1 \int_0^y \frac{3}{2}(x^2 + y^2)\, dx\, dy$$

(d) Find $P\left(Y > \frac{1}{2} \,\middle|\, Y > X\right)$. **Your answer should be an expression of parametrized double integrals, and you should not compute them.**

Solution: The denominator of the fraction should be your answer from part (c)—I won't take off points if you made a mistake there and repeat it here—and the numerator should be the integral of the density over the region of points (x, y) satisfying $y > 1/2$ and $y > x$. Here's a picture of the region:



The integral over this region could be either

$$\int_{1/2}^1 \int_0^y \frac{3}{2}(x^2 + y^2)\, dx\, dy$$

or

$$\int_0^{1/2} \int_{1/2}^1 \frac{3}{2}(x^2 + y^2)\, dy\, dx + \int_{1/2}^1 \int_x^1 \frac{3}{2}(x^2 + y^2)\, dy\, dx$$

So the correct answer is either of the two above integrals, divided by either of the two integrals from (c). Here's one valid answer:

$$\frac{\int_{1/2}^1 \int_0^y \frac{3}{2}(x^2 + y^2)\, dx\, dy}{\int_0^1 \int_0^y \frac{3}{2}(x^2 + y^2)\, dx\, dy}$$

[6 points] 6. Suppose that $X_1, X_2, \ldots$ are independent and identically distributed, with $E(X_i) = 6$ and $\mathrm{Var}(X_i) = 2$.

(a) Use Chebyshev's inequality to bound $P(|X_1 - 6| > 4)$.

Solution: Chebyshev's inequality gives

$$P\big(|X_1 - 6| > 4\big) \leq \frac{\mathrm{Var}(X_i)}{4^2} = \frac{1}{8}.$$

(b) Compute the mean and variance of $X_1 + \cdots + X_{10}$.

Solution: We have

$$E(X_1 + \cdots + X_{10}) = E(X_1) + \cdots + E(X_{10}) = 10(6) = 60.$$

That one would work regardless of whether the random variables were independent. For the variance,

$$\mathrm{Var}(X_1 + \cdots + X_{10}) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_{10}) = 10(2) = 20,$$

and we need the assumption of independence to get the equality of the variance of the sum and the sum of the variances.

7. Flip a fair coin four times. Let X be the number of tails in flips 1, 2, and 3. Let Y be the number of tails in flips 3 and 4.

[4 points] (a) Give the **marginal** distributions of X and Y. (Your answers should be named distributions and you should give their parameter(s).)

Solution: We have $X \sim \text{Bin}(3, 1/2)$ and $Y \sim \text{Bin}(2, 1/2)$.

[4 points] (b) Compute $P(X = 2, Y = 1)$.

Solution: The configurations producing $X = 2$, $Y = 1$ are 1101, 0110, 1010. So the probability is 3/16

[4 points] (c) Compute $P(X = 2 \mid Y = 1)$.

Solution:

$$P(X = 2 \mid Y = 1) = \frac{P(X = 2 \mid Y = 1)}{P(Y = 1)}$$
$$= \frac{3/16}{2(1/2)(1/2)} = \frac{3/16}{1/2} = \frac{3}{8}.$$

The value of $P(X = 2 \mid Y = 1)$ comes from your previous answer, the the value of $P(Y = 1)$ comes from $Y \sim \text{Bin}(2, 1/2)$.

[3 points] (d) Are X and Y independent or not? Justify your answer rigorously, with more than just intuition.

Solution: We need to find an example of x and y so that $P(X = x, Y = y) \neq P(X = x)\,P(Y = y)$. A simple one is $x = 3$ and $y = 0$: we have $P(X = 3, Y = 0) = 0$, since we can't have flips 1, 2, and 3 as tails but also have flips 3 and 4 as heads. But we know that $P(X = 3) > 0$ and $P(Y = 0) > 0$.

[1 point] (e) Which of the following is correct? (You do not need to justify your answer. You do not need to do any calculations to answer this question, if you use your intuition correctly.)

√ $\text{Cov}(X, Y) > 0$ ○ $\text{Cov}(X, Y) < 0$ ○ $\text{Cov}(X, Y) = 0$

[1 point] 8. A patient is tested for a rare genetic disease. You model the test probabilistically. Let D be the event that the person truly has the disease. Let N be the event that the person tests negative. Which of the following expressions represents the probability that a person who has tested positive truly has the disease?

○ $P(D \cap N^c)$ ○ $P(D \cup N^c)$ √ $P(D \mid N^c)$ ○ $P(N^c \mid D)$

[7 points] 9. Say whether the following statements are true or false.

(a) If events A and B are disjoint, then the event $A \cap B$ never occurs.
√ **True** ○ False

(b) If events A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$.
√ **True** ○ False

(c) If events A and B are independent, then $P(A \cap B) = P(A) + P(B)$.
○ True √ **False**

(d) It makes sense to talk about events A and B being disjoint, and it also makes sense to talk about random variables X and Y being disjoint.
○ True √ **False**

(e) It makes sense to talk about events A and B being independent, and it also makes sense to talk about random variables X and Y being independent.
√ **True** ○ False

(f) If random variables X and Y are independent, then their distributions are always discrete.
○ True √ **False**

(g) The expectation of any random variable is a positive number.
○ True √ **False**