Math 231, Final, Version A

May 22, 2024

Name: ____________________

I pledge that I have neither given nor received unauthorized assistance during this examination.

Signature:

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- You may use any sort of calculator, but not a phone or a computer.
- It is okay to leave a numerical answer like $\frac{39}{2} - (18 - e^2)$ unsimplified.
- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 10 problems on 10 pages.

Question	Points	Score
1	6	
2	4	
3	9	
4	9	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	88	

Good luck!

[6 points] 1. Find the derivatives of the following functions. **Do not simplify your solutions.**

(a) $f(x) = \sqrt{3x^2 + 2}$

Solution:

$$f'(x) = \frac{1}{2\sqrt{3x^2+2}}(6x)$$

(b) $g(x) = \dfrac{\sin(2x)}{4x+3}$

Solution:

$$g'(x) = \frac{(4x+3)2\cos(2x) - \sin(2x)(4)}{(4x+3)^2}$$

[4 points] 2. A car drives north on a straight road. After t hours, it has reached a distance of $15t^2$ miles from its starting point. What is its velocity at time $t = 2$?

Solution: We just need to find $s'(2)$, where $s(t) = 15t^2$. We compute $s'(t) = 30t$ and then $s'(2) = 60$ miles per hour.

[9 points] 3. Compute the following integrals.

(a) $\displaystyle\int xe^{4x^2+3}\,dx$

Solution: Use $u = 4x^2 + 3$, $du = 8x$. Then

$$\int xe^{4x^2+3}\,dx = \frac{1}{8}\int e^u\,du = \frac{1}{8}e^{4x^2+3} + C$$

(b) $\displaystyle\int_0^1 (3x^3 + 1)\,dx$

Solution:

$$\int_0^1 (3x^3 + 1)\,dx = \left(\frac{3}{4}x^3 + x\right)\Big|_0^1 = \frac{3}{4} + 1 = \frac{7}{4}.$$

(c) $\displaystyle\int \left(2\cos\left(\tfrac{x}{3}\right) + x\right)dx$

Solution:

$$6\sin\left(\tfrac{x}{3}\right) + \frac{x^2}{2} + C$$

[9 points] 4. Compute the following limits. If the limit does not exist, say so. (For a limit that diverges to infinity, it is fine either to say that it is infinite or that it doesn't exist.) Justify your answers by showing work or otherwise explaining your reasoning.

(a) $\lim_{x\to 5}(x^2 - x - 1)$

Solution:

$$\lim_{x\to 5}(x^2 - x - 1) = 5^2 - 5 - 1 = 19.$$

(b) $\lim_{x\to\infty} x^{-2}\ln x$

Solution: Apply L'Hôpital's rule:

$$\lim_{x\to\infty}\frac{\ln x}{x^2} = \lim_{x\to\infty}\frac{1/x}{x} = \lim_{x\to\infty}\frac{1}{x^2} = 0.$$

(c) $\lim_{x\to\infty}\frac{x^3 - 2x + 5}{2x^3 + 1}$

Solution:

$$\lim_{x\to\infty}\frac{x^3 - 2x + 5}{2x^3 + 1} = \lim_{x\to\infty}\frac{1 - \frac{2}{x^2} + \frac{5}{x^3}}{2 + \frac{1}{x^3}} = \frac{1}{2}.$$

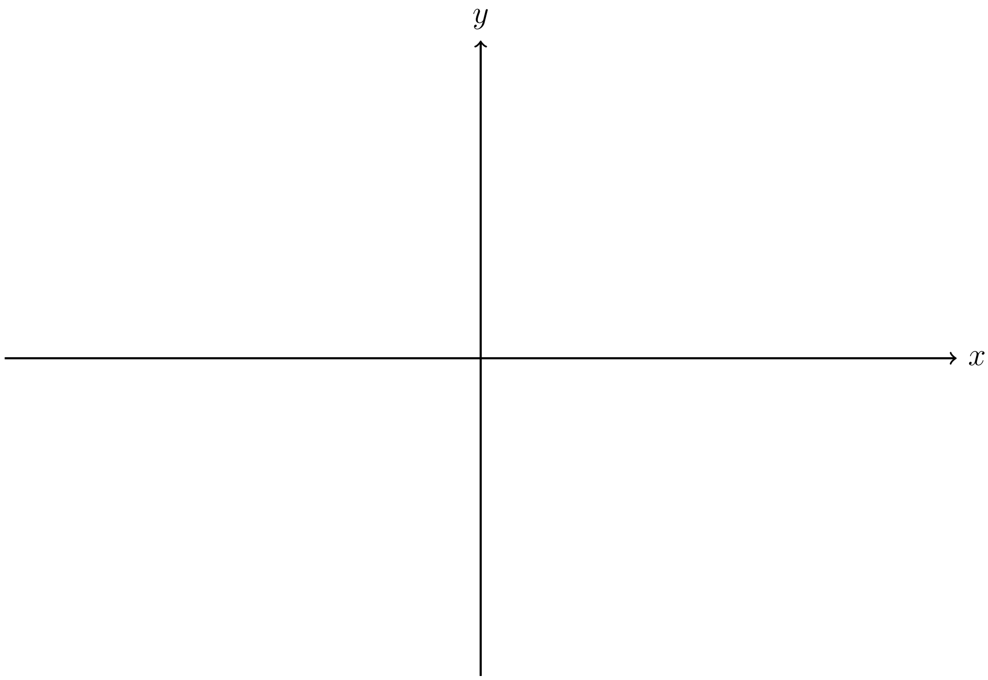
[10 points] 5. Let $f(x) = e^{3x}\sin(4x)$. Compute $f'(x)$ and $f''(x)$.

Solution:

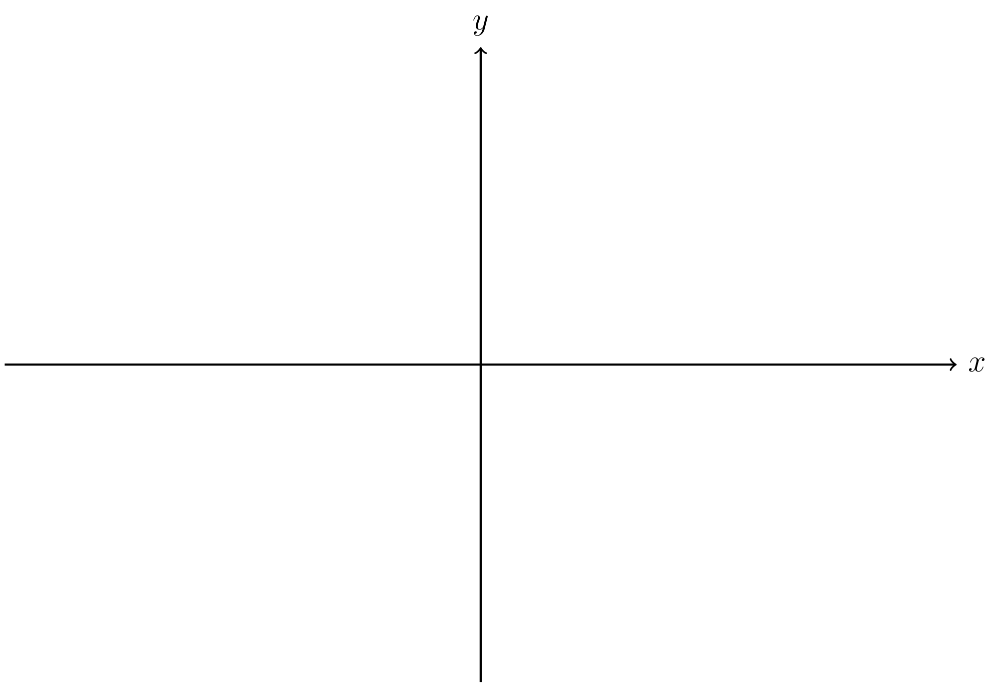
$$\begin{aligned}
f'(x) &= 3e^{3x}\sin(4x) + 4e^{3x}\cos(4x) \\
f''(x) &= 9e^{3x}\sin(4x) + 12e^{3x}\cos(4x) + 12e^{3x}\cos(4x) - 16e^{3x}\sin(4x) \\
&= -7e^{3x}\sin(4x) + 24e^{3x}\cos(4x)
\end{aligned}$$

[10 points] 6. (a) Sketch a graph of a function $f(x)$ with the following properties:

- it has a horizontal asymptote $y = 1$;
- its absolute maximum is 4;
- its absolute maximum occurs at $x = 2$.



(b) Sketch a graph of $f'(x)$. The graph you draw should be consistent with your sketch in part (a).



[10 points] 7. We need to enclose a rectangular field by a fence. A building is on one side of the field and so will not need any fencing. We have 500 feet of fencing to use for the other three sides of the field. Determine the dimensions of the field that give it the largest area.

Solution: We have a rectangle with sides of length x and y, but one of the sides (let's say y) doesn't count because it's the building. So we have a constraint $2x + y = 500$. Also we only consider x in the domain $[0, 250]$. We want to maximize $A = xy$. From the constraint, $y = 500 - 2x$, so

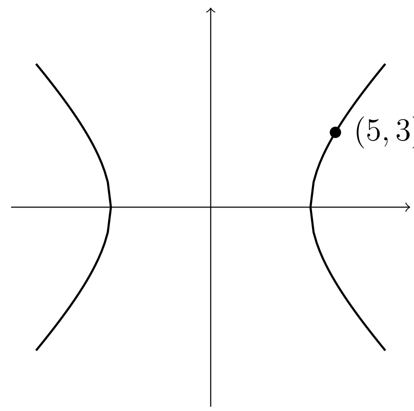
$$A(x) = x(500 - 2x) = 500x - 2x^2.$$

So

$$A'(x) = 500 - 4x$$

giving us a critical point at $x = 125$. At the endpoints 0 and 250 of the domain, we have $A(x) = 0$, so the maximum area occurs when $x = 125$. From the constraint, this gives $y = 250$. So the biggest area rectangle we can make has dimensions 125×250.

[10 points] 8. Consider the curve given by the solutions to $x^2 - y^2 = 16$, which forms a hyperbola containing the point $(5, 3)$.



(a) Find the equation of the tangent line to the curve at the point $(5, 3)$.

Solution: Differentiating both sides of the equation,

$$2x - 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{x}{y}.$$

Hence the slope of the curve at $(5, 3)$ is $\frac{5}{3}$. Using point-slope form, the tangent line at this point has equation

$$y - 3 = \frac{5}{3}(x - 5).$$

(b) Estimate y for the point $(5.1, y)$ near $(5, 3)$ on the curve.

Solution: One way to do this is to use the linear approximation equation

$$f(x + \delta x) \approx f(x) + f'(x)\delta x,$$

where $f(x)$ is the implicit function defined by the curve near $(5, 3)$, and $x = 5$ and $\delta x = .1$. We have $f(5) = 3$ and $f'(5) = \frac{5}{3}$, which gives

$$y = f(5.1) \approx 3 + \frac{5}{3}(.1) = 3.1666\ldots.$$

Alternatively, you can plug $x = 5.1$ into the tangent line obtained in (a) to get

$$y = 3 + \frac{5}{3}(5.1 - 5) = 3.1666\ldots.$$

[10 points] 9. A spherical balloon is filled with air at a rate of 4 cm^3/sec. How fast is the radius r of the balloon increasing when $r = 3$ cm?

Note: the volume of a sphere with radius r is equal to $\frac{4}{3}\pi r^3$.

Solution: Let r be the radius and V the volume of the balloon. We are given that $\frac{dV}{dt} = 4\ \text{cm}^3/\text{sec}$. We want to find $\frac{dr}{dt}$ when $r = 3$ cm. We relate V and r by $V = \frac{4}{3}\pi r^3$. Differentiating gives

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Now we substitute our known values to get

$$4 = 4\pi(3)^2 \frac{dr}{dt},$$

and we solve to get

$$\frac{dr}{dt} = \frac{1}{9\pi}\ \text{cm/sec}.$$

[10 points] 10. Suppose that the signs of $f'(x)$ and $f''(x)$ are as given below. The marks in the diagram indicate where the derivatives are equal to 0, and the + and − signs indicate where they are positive and negative.

On the axes below, sketch the graph of a function consistent with $f'(x)$ and $f''(x)$.