Math 311, Final
May 22, 2023

Name: ______________________________

I pledge that I have neither given nor received unauthorized assistance during this examination.

Signature:

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- No calculators or electronic devices are allowed.
- Unless the problem specifically says, you **do not** need to calculate the final answer. An answer that looks something like

$$\frac{\binom{6}{2}\binom{8}{5}}{\binom{23}{4}} \quad \text{or} \quad \frac{(12)(11)(10)+(11)(10)(9)}{14!}$$

  is entirely acceptable.
- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 10 problems on 8 pages.

Question	Points	Score
1	15	
2	9	
3	6	
4	11	
5	10	
6	10	
7	12	
8	5	
9	1	
10	4	
Total:	83	

Good luck!

[15 points] 1. Suppose we roll a 6-sided die over and over again.

In the following answers, you don't need to compute a final answer, and you may leave an expression like $\binom{13}{6}$ in that form. But you should not leave your answer as an infinite sum.

(a) What is the expected number of rolls to get a six?

Solution: This is the expected value of the Geo(1/6) distribution, which is 6.

(b) What is the probability that it takes exactly 5 rolls to get a six?

Solution: It's the probability of rolling 4 non-sixes followed by a six, or

$$\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right).$$

(c) What is the probability that it takes 5 or more rolls to get a six?

Solution: It's best to think of this as the probability that the first 4 rolls are non-sixes:

$$\left(\frac{5}{6}\right)^4$$

(d) What is the probability of getting 2 or fewer sixes in the first 10 rolls?

Solution: This is $P(S \leq 2)$ where $S \sim \text{Bin}(10, 1/6)$:

$$(5/6)^{10} + \binom{10}{1}(5/6)^9(1/6) + \binom{10}{2}(5/6)^8(1/6)^2.$$

(e) What is the probability that the sum of the first 2 rolls is 4 or less?

Solution: This is the probability of the first two rolls being either $(1,1)$, $(1,2)$, $(2,1)$, $(1,3)$, $(2,2)$, or $(3,1)$, which is a total of 6 of the 36 equally likely outcomes for the first two rolls. Thus the probability is $6/36 = 1/6$.

[9 points] 2. Suppose X and Y have joint density function

$$f(x,y) = \begin{cases} \frac{12}{7}(xy+y^2) & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal probability density function of X. **Fully compute all integrals. Be sure to clearly state where the p.d.f. is zero and where it's nonzero.**

Solution: For $0 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{12}{7}(xy+y^2)\,dy = \frac{12}{7}\left(xy^2/2 + y^3/3\right)\Big|_{y=0}^{y=1}$$
$$= \frac{12}{7}(x/2 + 1/3)$$

(b) Find $P(X < Y)$. **Set up integrals but do not compute them.**

Solution: We must integrate the density over the region $\{(x,y) \colon x < y\}$:

$$P(X<Y) = \int_0^1 \int_0^y \frac{12}{7}(xy+y^2)\,dx\,dy$$

(c) Compute $E(e^{XY^2})$. **Set up integrals but do not compute them.**

Solution:

$$\int_0^1 \int_0^1 e^{xy^2} \frac{12}{7}(xy+y^2)\,dx\,dy.$$

[6 points] 3. Let $X \sim \text{Unif}[0,2]$. Let $Y = 1/X$. Find the probability density function of Y. If the density is zero outside of some region, be sure to clearly state this.

Solution: We compute the c.d.f. of Y. Since X is between 0 and 2, the random variable Y is between $1/2$ and ∞. For $x \geq 1/2$,

$$F_Y(x) = P(Y \leq x) = P(X \geq 1/x) = \frac{2 - 1/x}{2} = 1 - \frac{1}{2x}.$$

Now we compute

$$F_Y'(x) = \frac{1}{4x^2}.$$

So, the density is

$$f(x) = \begin{cases} \frac{1}{4x^2} & \text{if } x \geq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

4. There are 5 types of sandwiches for sale at a shop. For 10 days, you go to the shop for lunch and order one of the 5 sandwiches uniformly at random, independently of previous days.

[4 points] (a) What is the probability that you order the falafel sandwich sometime during the 10 days?

> **Solution:** It's easier to compute the probability of never ordering the falafel sandwich: $(4/5)^{10}$. Thus the probability that you ever order the falafel sandwich is $1-(4/5)^{10}$.

[4 points] (b) Let X be the number of different sandwiches out of the 5 types you try in the 10 days. Find $E(X)$.

> **Solution:** Write X as $B_1+B_2+B_3+B_4+B_5$ where B_i is an indicator on ever ordering sandwich type i. From the calculation before about the falafel sandwich, $E(B_i)=1-(4/5)^{10}$. By linearity of expectation,
>
> $$E(X)=5(1-(4/5)^{10})\approx 4.46.$$

[3 points] (c) For $i=1,\ldots,5$, let X_i be the number of times you order sandwich type i in the 10 days. Are $X_1,\ldots,X_5$ independent? Justify your answer.

> **Solution:** No, they are not independent. For example,
>
> $$P(X_1=3,X_2=3,X_3=3,X_4=3,X_5=3)=0,$$
>
> since the total number of sandwiches ordered has to be 10. But this is not equal to $P(X_1=3)P(X_2=3)P(X_3=3)P(X_4=3)P(X_5=3)$, since each of those probabilities is nonzero.

5. Suppose that $X \sim \mathrm{Poi}(100)$, and recall that $E(X) = 100$ and $\mathrm{Var}(X) = 100$.

[3 points] (a) Give a bound on $P(X \geq 120)$ using Markov's inequality.

Solution:

$$P(X \geq 120) \leq \frac{E(X)}{120} = \frac{5}{6}.$$

[3 points] (b) Give a bound on $P(X \geq 120)$ using Chebyshev's inequality.

Solution:

$$P(X \geq 120) = P(X - 100 \geq 20) \leq \frac{\mathrm{Var}(X)}{20^2} = \frac{1}{4}.$$

[4 points] (c) Recall that if $X_1, \ldots, X_{100}$ are independent and $X_i \sim \mathrm{Poi}(1)$, then $X_1 + \cdots + X_{100} \sim \mathrm{Poi}(100)$. Using the central limit theorem, estimate

$$P(X_1 + \cdots + X_{100} \geq 120),$$

to get another estimate of $P(X \geq 120)$. **Write your answer in terms of $\Phi(x)$, the cumulative distribution function of the standard normal distribution.**

Solution: The sum $X = X_1 + \cdots + X_{100}$ has mean 100 and variance 100. By the central limit theorem,

$$\frac{X - 100}{\sqrt{100}}$$

approximately has the $N(0, 1)$ distribution. So,

$$P(X \geq 120) = P\left(\frac{X_1 + \cdots + X_{100} - 100}{10} \geq \frac{120 - 100}{10}\right) \approx 1 - \Phi(2) \approx 0.023.$$

6. Customers arrive at a store. After a customer arrives, the number of minutes until the next customer arrives is random with distribution Exp(1). The times between customer arrivals are all independent.

   A customer arrives. Let X be the number of minutes from now until *two* more customers arrive.

[3 points] (a) What is $E(X)$?

Solution: Let T_1 be the time from now until the next customer arrives, and then let T_2 be the time from then to the next customer. Then $X = T_1 + T_2$. We are told that $T_1, T_2 \sim \mathrm{Exp}(1)$ and that T_1 and T_2 are independent. We have $E(T_i) = 1$ by our knowledge of the exponential distribution. By linearity of expectation,

$$E(X) = E(T_1) + E(T_2) = 1 + 1 = 2.$$

[3 points] (b) What is $\mathrm{Var}(X)$?

Solution: We have $\mathrm{Var}(T_i) = 1$ by our knowledge of the exponential distribution (or we could compute it). Since T_1 and T_2 are independent,

$$\mathrm{Var}(X) = \mathrm{Var}(T_1) + \mathrm{Var}(T_2) = 2.$$

[4 points] (c) What is the probability density function of X?

Solution: We use the convolution formula:

$$f_X(x) = \int_{-\infty}^{\infty} f_{T_1}(u) f_{T_2}(x-u)\,du = \int_0^x e^{-u}e^{-(x-u)}\,du = \int_0^x e^{-x}\,dx = xe^{-x}.$$

7. You roll two 4-sided dice. Let U be the minimum and V be the maximum of the two rolls.

[6 points] (a) Find the joint probability mass function of U and V. It is easiest to show it as a table, though you can express it any way you want so long as it's clear.

Solution: We must have $V \geq U$, so put probability 0 on each (u, v) pair with $v < u$. If $u = v$, then both rolls must be equal to this value; there is only one way to achieve this roll out of the 16 possible rolls, for a probability of $1/16$. If $u < v$, then one roll is u and the other is v, and there are two ways to achieve this, making the probability $1/8$. So the table is:

		V 1	2	3	4
	1	1/16	1/8	1/8	1/8
U	2	0	1/16	1/8	1/8
	3	0	0	1/16	1/8
	4	0	0	0	1/16

[2 points] (b) What is $P(U = V)$?

Solution: Reading the answer off the table, it's

$$\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{4}.$$

[2 points] (c) What is $P(V = 4 \mid U = 3)$?

Solution:

$$\frac{1/8}{1/16 + 1/8} = \frac{2}{3}.$$

[2 points] (d) Are U and V independent? Justify your answer.

Solution: No, definitely not. For example $P(U = 4, V = 1) = 0$, but $P(U = 4) \neq 0$ and $P(V = 1) \neq 0$.

[5 points] 8. You have an urn with balls numbered $1, \ldots, 10$. You reach in and pull out three balls. You don't consider your three chosen balls to have any order.

(a) The size of the most natural sample space for the experiment is

○ 10^3 √ $\binom{10}{3}$ ○ $10 \cdot 9 \cdot 8$ ○ $3!$

(b) Let A be the event that your sample includes ball 1 and B the event that your sample contains ball 2. Then A and B are

√ **not disjoint** ○ disjoint

and A and B are

√ **not independent** ○ independent

(c) Let A be the event that your sample includes balls 1 and 2, and let B be the event that your sample contains ball 3 and 4. Then A and B are

○ not disjoint √ **disjoint**

and A and B are

√ **not independent** ○ independent

[1 point] 9. A patient is tested for a rare genetic disease. You model the test probabilistically. Let D be the event that the person truly has the disease. Let N be the event that the person tests negative. Which of the following expressions represents the probability that a person who has tested positive truly has the disease?

√ $P(D \mid N^c)$ ○ $P(N^c \mid D)$ ○ $P(D \cap N^c)$ ○ $P(D \cup N^c)$

[4 points] 10. Suppose that X is a continuous random variable with probability density function $f(x)$.

(a) The range of possible values of X is given by the range of y-values of $f(x)$.

○ True √ **False** ○ It depends on $f(x)$

(b) The range of possible values of X is given by the set of x-values for which $f(x) > 0$.

√ **True** ○ False ○ It depends on $f(x)$

(c) For any real number k, it holds that $P(X = k) = 0$.

√ **True** ○ False ○ It depends on $f(x)$

(d) For any real numbers a and b with $a < b$, it holds that $P(a < X < b) = P(a \leq X \leq b)$.

√ **True** ○ False ○ It depends on $f(x)$