Math 232, Final Name: ______________________

December 18, 2023

I pledge that I have neither given nor received unauthorized assistance during this examination.

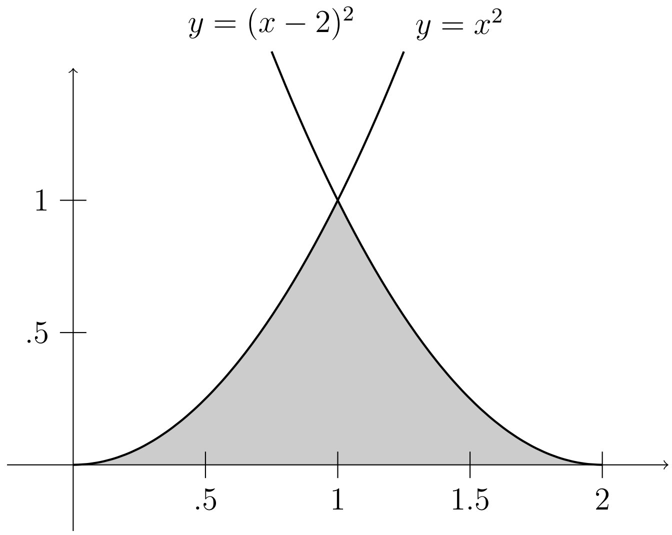
Signature:

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- You may use a scientific calculator, but not a graphing calculator or phone.
- It is okay to leave a numerical answer like $\frac{39}{2} - (18 - e^2)$ unsimplified.
- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 9 problems on 12 pages.

Question	Points	Score
1	10	
2	8	
3	8	
4	8	
5	8	
6	11	
7	12	
8	6	
9	6	
Total:	77	

Good luck!

[10 points] 1. A region is bounded by the x-axis and the curves $y = x^2$ and $y = (x-2)^2$:



The region is rotated around the y-axis to form a solid. (Please read carefully: **y-axis**.)

(a) Set up **but do not compute** an expression to find the volume of the solid using the disc/washer method.

Solution: For a rotation around the y-axis using the disc/washer method, we integrate with respect to y. We need to solve for x in terms of y for both of the curves. For the curve $y = x^2$, we have $x = \pm\sqrt{y}$. Since we want the part of the curve with positive x-value, it's $x = \sqrt{y}$. For the curve $y = (x-2)^2$, we have $x = 2 \pm \sqrt{y}$. Since we want the part of the curve to the left of $x = 2$, it's $x = 2 - \sqrt{y}$.

Now, we integrate from $y = 0$ to $y = 1$, adding up a washer at height y that has inner radius $\sqrt{y}$ and outer radius $2 - \sqrt{y}$. So the integral is

$$\pi \int_0^1 \left((2-\sqrt{y})^2 - (\sqrt{y})^2\right) dy.$$

(b) Set up **but do not compute** an expression to find the volume of the solid using the cylindrical shells method.

Solution: For shells and a rotation around the y-axis, we integrate with respect to x. The height of the shell is given by the height of the curve, which is x^2 for x between 0 and 1 and $(x-2)^2$ for x between 1 and 2. The radius of the shell is just x. So we have to break the integral in two to get

$$\int_0^1 2\pi x(x^2)\,dx + \int_1^2 2\pi x(x-2)^2\,dx.$$

[8 points] 2. State whether the following series converge or diverge. If the series converges, **compute its sum**. If not, explain why it diverges. If you apply a test, you must give all details of the test to get full credit. (For example, for the comparison test, say what series you're comparing to.)

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{2^n}}$

Solution: Since $\sqrt{2^n} = 2^{n/2}$, this series is $1 + 2^{-1/2} + 2^{-1} + 2^{-3/2} + \cdots$, which is geometric with first term 1 and multiplier $2^{-1/2}$, which is less than 1. So its sum is

$$\frac{1}{1 - 2^{-1/2}}.$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n+1}$

Solution: The limit of the nth term is

$$\lim_{n\to\infty} \frac{n+1}{2n+1} = \lim_{n\to\infty} \frac{1 + 1/n}{2 + 1/n} = \frac{1}{2}.$$

Since this is not equal to 0, the series diverges.

[8 points] 3. State whether the following series converge or diverge. Explain your answer. If you apply a test, you must give all details of the test to get full credit. (For example, for the comparison test, say what series you're comparing to.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5^n}$

Solution: The ratio of $(n+1)$th and nth terms is

$$\frac{n+1^2}{5^{n+1}}\frac{5^n}{n^2} = \frac{1}{5}\left(\frac{n+1}{n}\right)^2.$$

This has limit $1/5$ as $n \to \infty$ which is less than 1, so the series converges by the ratio test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

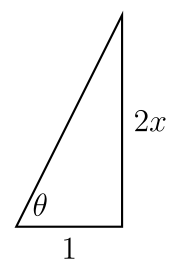
Solution: The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test, and $\frac{1}{n^2+1} \leq \frac{1}{n^2}$. By the comparison test, this sum converges too.

[8 points] 4. Compute $\displaystyle\int \frac{1}{(4x^2+1)^{3/2}}\,dx$.

Solution: We use trigonometric substitution. Set $x = \frac{1}{2}\tan\theta$. Then $dx = \frac{1}{2}\sec^2\theta\,d\theta$, and

$$\begin{aligned}\int \frac{1}{(4x^2+1)^{3/2}}\,dx &= \int \frac{\sec^2\theta}{2(\tan^2\theta+1)^{3/2}}\,d\theta \\ &= \int \frac{\sec^2\theta}{2\sec^3\theta}\,d\theta \\ &= \frac{1}{2}\int \cos\theta\,d\theta = \frac{1}{2}\sin\theta + C.\end{aligned}$$

Now we need to substitute back to replace θ with an expression in terms of x. We know that $\tan\theta = 2x$. Given this, what is $\sin\theta$? To answer this question, we draw a triangle based on the information $\tan\theta = 2x$:



The hypotenuse is $\sqrt{4x^2+1}$ by the Pythagorean theorem. So, $\sin\theta = 2x/\sqrt{4x^2+1}$. Plugging this back in gives

$$\int \frac{1}{(4x^2+1)^{3/2}}\,dx = \frac{x}{\sqrt{4x^2+1}} + C.$$

[8 points] 5. Find the Taylor series of the following functions centered at 0. You do not need to determine the interval of convergence.

(a) $f(x) = xe^{-x^2}$

Solution: Start with the known series $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$. Now substitute x with $-x^2$ and multiply by x to get

$$\begin{aligned} f(x) &= x - x^3 + \frac{x^5}{2!} - \frac{x^7}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!}. \end{aligned}$$

(b) $f(x) = \dfrac{x}{1-2x}$

Solution: This expression is the sum of a geometric series with first term x and multiplier $2x$. So,

$$\begin{aligned} f(x) &= x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + \cdots \\ &= \sum_{n=0}^{\infty} 2^n x^{n+1}. \end{aligned}$$

6. A particle at time t is at position $\big(x(t),\, y(t)\big)$ where

$$x(t) = \sin(2t)\cos(t),$$
$$y(t) = \sin(2t)\sin(t).$$

[3 points] (a) At time 0, the particle is at position $(0,0)$. When is the next time that it returns to $(0,0)$?

Solution: The next time that $x(t) = 0$ is when $t = \pi/2$, since then $\sin(2t) = 0$. And the same is true for the next time $y(t) = 0$. So the next time the particle arrives back at $(0,0)$ is $t = \pi/2$.

[4 points] (b) The particle sketches out a curve. What is the slope of the tangent line to this curve at the particle's location at time $\pi/4$?

Solution: We have

$$x'(t) = 2\cos(2t)\cos(t) - \sin(2t)\sin(t),$$
$$y'(t) = 2\cos(2t)\sin(t) + \sin(2t)\cos(t).$$

At time $t = \pi/4$, this gives

$$x'(\pi/4) = 2(0)\frac{\sqrt{2}}{2} - (1)\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2},$$
$$y'(\pi/4) = 2(0)\frac{\sqrt{2}}{2} + (1)\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}.$$

The slope of the tangent line is $y'(\pi/4)/x'(\pi/4)$, which is -1.

[4 points] (c) Set up **but do not compute** an integral to find the total distance traveled by the particle from time 0 to time $\pi/2$.

Solution:

$$\int_0^{\pi/2} \sqrt{x'(t)^2 + y'(t)^2}\,dt$$
$$= \int_0^{\pi/2} \sqrt{(\cos(2t)\cos(t) - \sin(2t)\sin(t))^2 + (2\cos(2t)\sin(t) + \sin(2t)\cos(t))^2}\,dt.$$

[12 points] 7. Compute the following integrals. If they do not converge, say so and explain why.

(a) $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$

Solution:

$$\int_2^\infty \frac{1}{x^2}\,dx = \lim_{R\to\infty} -\frac{1}{x}\bigg|_2^R = \lim_{R\to\infty}\left(-\frac{1}{R}+\frac{1}{2}\right) = \frac{1}{2}.$$

(b) $\displaystyle\int \frac{x+1}{(x-1)^2}\,dx$

Solution: Here are two possible solutions. First, do the substitution $u = x-1$, which turns the integral into

$$\int \frac{u+2}{u^2}\,du = \int \left(u^{-1}+2u^{-2}\right)du = \ln|u| - 2u^{-1} + C = \ln|x-1| - \frac{2}{x-1} + C.$$

Alternatively, use partial fractions, rewriting the integrand as

$$\frac{x+1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

Clearing fractions gives

$$x+1 = A(x-1)+B.$$

We can substitute $x=1$ to get $B=2$. And then we can substitute $x=0$ to get $1=-A+2$ which gives $A=1$. So,

$$\begin{aligned}\int \frac{x+1}{(x-1)^2}\,dx &= \int\left(\frac{1}{x-1}+\frac{2}{(x-1)^2}\right)dx \\ &= \ln|x-1| - \frac{2}{x-1} + C.\end{aligned}$$

(c) $\displaystyle\int x\ln x\,dx$

Solution: Integration by parts with $u=\ln x$, $dv = x\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = x^2/2$, and

$$\begin{aligned}\int x\ln x\,dx = uv - \int v\,du &= \frac{x^2\ln x}{2} - \int \frac{x^2}{2}\frac{1}{x}\,dx \\ &= \frac{x^2\ln x}{2} - \frac{x^2}{4} + C\end{aligned}$$

[6 points] 8. (a) Convert the point $(x, y) = (-2\sqrt{3}, 2)$ to polar coordinates (r, θ).

Solution: We know that $\tan\theta = -1/\sqrt{3}$. This is consistent with $\theta = -\pi/6$ or $\theta = 5\pi/6$. Since the point is in the second quadrant, it's $\theta = 5\pi/6$. And $r = \sqrt{(-2\sqrt{3})^2 + 2^2} = 4$.

(b) Convert the point $(r, \theta) = (12, -\pi/4)$ to rectangular coordinates (x, y).

Solution: It's just

$$(r\cos\theta, r\sin\theta) = (6\sqrt{2}, -6\sqrt{2}).$$

[6 points] 9. For which values of x does the following power series converge?

$$\sum_{n=0}^{\infty} \frac{nx^n}{2^n}$$

Solution: The ratio of the $(n+1)$th and nth terms is

$$\frac{(n+1)x^{n+1}}{2^{n+1}} \frac{2^n}{nx^n} = \frac{n+1}{2n} x.$$

The limit of the absolute value of this expression as $n \to \infty$ is $|x|/2$. By the ratio test, the series converges when $|x|/2 < 1$, which occurs when $-2 < x < 2$. Also by the ratio test, the series diverges when $|x|/2 > 1$, which occurs when $x < -2$ or $x > 2$.

We still need to say what happens when $x = -2$ or $x = 2$. The series in these two cases are

$$\sum_{n=0}^{\infty} (-1)^n n \quad \text{and} \quad \sum_{n=0}^{\infty} n.$$

Both of these series diverge, since the limit of the their nth terms don't converge to 0. So the series converges only for $-2 < x < 2$.

TABLE OF TRIGONOMETRIC INTEGRALS

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C = \frac{x}{2} - \frac{1}{2}\sin x\cos x + C \qquad \mathbf{3}$$

$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C = \frac{x}{2} + \frac{1}{2}\sin x\cos x + C \qquad \mathbf{4}$$

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx \qquad \mathbf{5}$$

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx \qquad \mathbf{6}$$

$$\int \tan x\,dx = \ln|\sec x| + C = -\ln|\cos x| + C \qquad \mathbf{7}$$

$$\int \tan^m x\,dx = \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x\,dx \qquad \mathbf{8}$$

$$\int \cot x\,dx = -\ln|\csc x| + C = \ln|\sin x| + C \qquad \mathbf{9}$$

$$\int \cot^m x\,dx = -\frac{\cot^{m-1} x}{m-1} - \int \cot^{m-2} x\,dx \qquad \mathbf{10}$$

$$\int \sec x\,dx = \ln\left|\sec x + \tan x\right| + C \qquad \mathbf{11}$$

$$\int \sec^m x\,dx = \frac{\tan x\sec^{m-2} x}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2} x\,dx \qquad \mathbf{12}$$

$$\int \csc x\,dx = \ln\left|\csc x - \cot x\right| + C \qquad \mathbf{13}$$

$$\int \csc^m x\,dx = -\frac{\cot x\csc^{m-2} x}{m-1} + \frac{m-2}{m-1}\int \csc^{m-2} x\,dx \qquad \mathbf{14}$$

$$\int \sin mx\sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + C \quad (m \neq \pm n) \qquad \mathbf{15}$$

$$\int \sin mx\cos nx\,dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} + C \quad (m \neq \pm n) \qquad \mathbf{16}$$

$$\int \cos mx\cos nx\,dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + C \quad (m \neq \pm n) \qquad \mathbf{17}$$