Math 311, Final Name: ______________________________

May 18, 2020

I pledge that I have neither given nor received unauthorized assistance during this examination.

Signature:

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- No calculators or electronic devices are allowed.
- Unless the problem specifically says, you **do not** need to calculate the final answer. An answer that looks something like

$$\frac{\binom{6}{2}\binom{8}{5}}{\binom{23}{4}} \quad \text{or} \quad \frac{(12)(11)(10)+(11)(10)(9)}{14!}$$

  is entirely acceptable.
- You may consult your notes, the textbook, and any other resources that are specifically for this class (Webwork, my videos, etc.).
- You may not consult other people, outside resources (other textbooks or things on the general internet).
- There are 7 problems on 7 pages.

Question	Points	Score
1	22	
2	12	
3	16	
4	15	
5	8	
6	12	
7	15	
Total:	100	

Good luck!

1. A sidewalk is lined with elm trees. Each tree independently has a 1/10 probability of getting Dutch elm disease. In order from the first tree on sidewalk onward, number the trees $1, 2, \ldots$.

   Let X be the number of trees out of the first 13 that have Dutch elm disease. Let Y be the number of the first tree on the block with Dutch elm disease.

   In the following answers, you dont need to compute a final answer, and you may leave an expression like $\binom{5}{3}$ in that form. But you should not leave your answer as an infinite sum.

[4 points] (a) Give the expectation and variance of X.

Solution: Since $X \sim \text{Bin}(13, 1/10)$,

$$\begin{aligned} E(X) &= (13)(1/10), \\ \text{Var}(X) &= (13)(1/10)(1-1/10). \end{aligned}$$

[3 points] (b) What is $P(X = 3)$?

Solution:

$$P(X = 3) = \binom{13}{3}(1/10)^3(9/10)^{10}$$

[3 points] (c) What is $P(X \le 2)$?

Solution:

$$P(X \le 2) = \binom{13}{2}(1/10)^2(9/10)^{11} + \binom{13}{1}(1/10)(9/10)^{12} + (9/10)^{13}$$

[3 points] (d) What is the probability that the first three trees on the block all have Dutch elm disease?

Solution: Since each of these trees has probability 1/10 of having Dutch elm disease, the probability that all three do is $(1/10)^3$.

[3 points] (e) What is $P(Y = 2)$?

Solution:

$$P(Y = 2) = (9/10)^1(1/10)$$

[3 points] (f) What is $P(Y > 2)$?

Solution:

$$P(Y > 2) = (9/10)^2$$

[3 points] (g) What is $P(X = 1 \mid Y = 1)$?

Solution: The information $Y = 1$ tells us that the first tree has Dutch elm disease. Given this, the event $X = 1$ occurs if none of the next 12 trees has Dutch elm disease. Hence,

$$P(X = 1 \mid Y = 1) = (9/10)^{12}.$$

[12 points] 2. A hotel building has 21 stories. This evening, 5 guests get into an elevator on floor #1 and press the button to go to their rooms. Assume that their rooms are independently chosen from floors $2, \dots, 21$, uniformly at random.

(a) What is the probability that there exist two guests in the elevator whose floors are the same?

Solution: This is the birthday paradox. The probability that all 5 guests are on different floors is

$$\left(\frac{20}{20}\right)\left(\frac{19}{20}\right)\cdots\left(\frac{16}{20}\right)$$

The probability that there exist two guests in the elevator whose floors are the same is one minus this.

(b) Let X be the number of different floors from $2, \dots, 21$ the elevator will stop at. Find $E(X)$.

Solution: Using the indicator method, write X as

$$X = J_2 + \cdots + J_{21},$$

where J_i is 1 if the button for floor i is pressed and 0 if not. We compute

$$\begin{aligned} E(J_i) &= P(\text{someone goes to floor } i) \\ &= 1 - P(\text{nobody goes to floor } i) \\ &= 1 - \left(\frac{19}{20}\right)^5. \end{aligned}$$

So,

$$\begin{aligned} E(X) &= E(J_2) + \cdots + E(J_{21}) \\ &= 20\left(1 - \left(\frac{19}{20}\right)^5\right) \end{aligned}$$

3. A gas station serves a random number of customers each day. On average, there are 700 customers a day, with a variance of 30.

[4 points] (a) Based on this information, give the best available bound on the probability that there are 745 or more customers tomorrow. **Note:** Give an exact answer. Do not use a calculator. You can leave your answer as an unevaluated expression.

Solution: Let Y be the number of customers tomorrow. We're given that $E(Y) = 700$ and $\mathrm{Var}(Y) = 30$. By Chebyshev's inequality,

$$P(Y \geq 745) = P(Y - 700 \geq 45) \leq \frac{30}{45^2}$$

[6 points] (b) Let X be the total number of customers in the month of April, which has 30 days. Assume that the numbers of customers on different days are independent. Find $E(X)$ and $\mathrm{Var}(X)$.

Solution: In this problem, $X = Y_1 + \cdots + Y_{30}$, where Y_i has mean 700 and variance 30. By linearity, we have $E(X) = 30(700) = 21000$. Since $Y_1, \ldots, Y_{30}$ are independent, the variances add as well, and $\mathrm{Var}(X) = 30(30) = 900$.

[6 points] (c) Use the central limit theorem to estimate $P(X > 20955)$. Give your answer in terms of $\Phi(x)$, the cumulative distribution function of the standard normal distribution.

Solution: As we found in the last problem, the mean and standard deviation of X are 21000 and $30 = \sqrt{900}$. By the CLT, the random variable $\frac{X-21000}{30}$ approximately has the standard normal distribution. So,

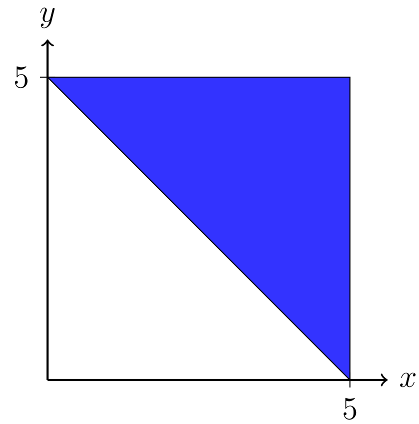
$$P(X > 20955) = P\left(\frac{X - 21000}{30} > \frac{20955 - 21000}{30}\right) = 1 - \Phi(-1.5).$$

[15 points] 4. Let X and Y have joint probability density function

$$f(x,y) = \begin{cases} \frac{42}{265625}x^2y^3 & \text{if } 0 \leq x, y \leq 5 \text{ and } y \geq 5 - x, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Set up **but do not compute** an integral to find $E(XY)$.

Solution: The region defined by $0 \leq x, y \leq 5$ and $y \geq 5 - x$ looks like this:



We have to integrate xy times the density over this region:

$$E(XY) = \int_0^5 \int_{5-y}^5 \frac{42}{265625} x^3 y^4 \, dx \, dy$$

or

$$E(XY) = \int_0^5 \int_{5-x}^5 \frac{42}{265625} x^3 y^4 \, dy \, dx$$

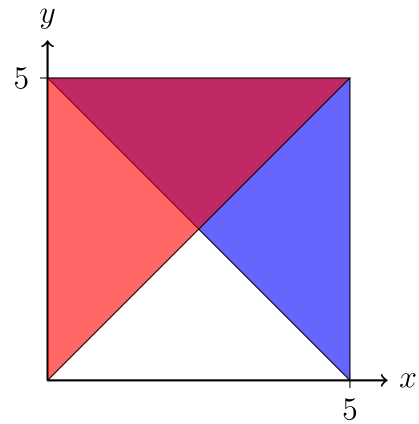
(b) Let $f_X(x)$ be the marginal probability density function of X. Set up **but do not compute** an integral to find $f_X(x)$ when $0 \leq x \leq 5$.

Solution:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \int_{5-x}^5 \frac{42}{265625} x^2 y^3 \, dy$$

(c) Set up **but do not compute** an integral to find $P(Y \geq X)$.

Solution: We need to integrate the density over the purple region in this picture, the intersection of the blue and red regions:



This is given either by

$$\int_{5/2}^{5}\int_{5-y}^{y} \frac{42}{265625} x^2 y^3 \, dx \, dy$$

or

$$\int_{0}^{5/2}\int_{5-x}^{5} \frac{42}{265625} x^2 y^3 \, dy \, dx + \int_{5/2}^{5}\int_{x}^{5} \frac{42}{265625} x^2 y^3 \, dy \, dx$$

[8 points] 5. Let $X \sim \text{Exp}(2)$ and let $Y = X^2$. Find the probability density function of Y. If the density is zero outside of some region, be sure to clearly state this.

Solution: The density of X is $f(x) = 2e^{-2x}$ for $x \geq 0$. The cdf of Y is

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = \int_0^{\sqrt{y}} 2e^{-2x} \, dx$$
$$= -e^{-2x}\Big|_0^{\sqrt{y}} = 1 - e^{-2\sqrt{y}}$$

for $y \geq 0$, with $F_Y(y) = 0$ for $y < 0$. Then the density of Y is

$$f_Y(y) = F_Y'(y) = e^{-2\sqrt{y}} \frac{2}{2\sqrt{y}} = \frac{1e^{-2\sqrt{y}}}{\sqrt{y}}$$

for $y \geq 0$ (with the density equal to 0 for $y < 0$).

[12 points] 6. You have 4 pennies, 3 nickels, and 3 dimes in your pocket, and you reach into your pocket and put three coins at random in your hand. **It's fine to leave uncomputed expressions like $\binom{7}{3}$ in your answers.**

(a) What is the probability that you have one penny, one nickel, and one dime in your hand?

> **Solution:** Here, we are sampling without replacement, and we can consider order not to matter. Thus we are choosing an outcome uniformly at random from a solution space of size $\binom{10}{3}$. The number of ways to choose one penny, one nickel, and one dime is $4 \cdot 3 \cdot 3$. So the probability is
>
> $$\frac{4 \cdot 3 \cdot 3}{\binom{10}{3}} = \frac{3}{10}.$$

(b) What is the expected number of dimes in your hand?

> **Solution:** Here are two different solutions:
> Let $J_1, \ldots, J_3$ be indicators on including the various dimes in your hand. That is, J_i is 1 if a dime $\#i$ is in your hand and 0 if not. Since there are $\binom{10-1}{2} = 36$ ways to choose the other two coins in your hand, the probability of getting dime $\#i$ in your hand is $36/\binom{10}{3} = \frac{3}{10}$. So, $E(J_i) = \frac{3}{10}$. The number of dimes in your hand is $X = J_1 + \cdots + J_3$, and so by linearity of expectation
>
> $$E(X) = E(J_1) + \cdots + E(J_3) = 3\left(\frac{3}{10}\right).$$
>
> Alternatively, change the sample space to the one for sampling without replacement where order matters. Let J_1, J_2, J_3 be indicators on the three different choices being dimes. The probability that the first choice is a dime is $3/10$, so $E(J_1) = 3/10$. Since there's nothing truly different about the second and third choices, $E(J_2)$ and $E(J_3)$ are also equal to this. (This is clear by intuition, though to really justify it we'd need to use a section of the textbook that we didn't cover on the topic of *exchangeability*.) The number of dimes in your hand is $X = J_1 + J_2 + J_3$, and by linearity of expectation
>
> $$E(X) = E(J_1) + E(J_2) + E(J_3) = 3\left(\frac{3}{10}\right).$$

[15 points] 7. Let X and Y have joint probability mass function given by the following table

		Y = 1	2	3
	1	0.05	0.25	0.1
X	2	0	0.1	0.15
	3	0.05	0.1	0.2

(a) Find the marginal probability mass function of X.

Solution: By summing the rows of the joint pmf, we get

$$p_X(1) = 0.40$$
$$p_X(2) = 0.25$$
$$p_X(3) = 0.35$$

(b) Find $P(X = 1, Y = 1)$.

Solution: This is just the $X = 1$, $Y = 1$ entry of the table: 0.05

(c) Find $P(X = 1 \text{ or } Y = 1)$.

Solution: This is given by summing the five entries in the first row and column:

$$0.05 + 0.25 + 0.1 + 0 + 0.05 = 0.45$$

(d) Find $P(Y = 1 \mid X = 1)$.

Solution:

$$P(Y = 1 \mid X = 1) = \frac{P(X = 1, Y = 1)}{P(X = 1)} = \frac{0.05}{0.40}$$

(e) Are X and Y independent? Justify your answer.

Solution: They are not independent. To justify this, you just need to show one example where $P(X = j, Y = k) \neq P(X = j)P(Y = k)$. For example, you could point out that $P(X = 2, Y = 1) = 0$ even though $P(X = 2) \neq 0$ and $P(Y = 1) \neq 0$.