

*I pledge that I have neither given nor received
unauthorized assistance during this examination.*

Signature:

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- No calculators or electronic devices are allowed.
- Unless the problem specifically says, you **do not** need to calculate the final answer. An answer that looks something like

$$\frac{\binom{6}{2}\binom{8}{5}}{\binom{23}{4}} \quad \text{or} \quad \frac{(12)(11)(10) + (11)(10)(9)}{14!}$$

is entirely acceptable.

- You may bring two pages of two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 8 problems on 9 pages.

Question	Points	Score
1	20	
2	16	
3	8	
4	12	
5	12	
6	12	
7	8	
8	12	
Total:	100	

Good luck!

[20 points] 1. When somebody sends you a text message, you do one of the following, independent of all previous text messages:

- completely ignore the message with probability $1/6$;
- read the message but don't respond with probability $1/3$;
- respond to the message with probability $1/2$.

In the following answers, you don't need to compute a final answer, and you may leave an expression like $\binom{13}{6}$ in that form. But you should not leave your answer as an infinite sum.

(a) What is the probability that you completely ignore the next 10 texts?

Solution: $(1/6)^{10}$

(b) What is the probability that you completely ignore exactly 7 out of the next 10 texts?

Solution: $\binom{10}{7} \left(\frac{1}{6}\right)^7 \left(\frac{5}{6}\right)^3$

(c) What is the expected number of completely ignored texts out of the next 10?

Solution: $10 \cdot \frac{1}{6} = \frac{5}{3}$

- (d) What is the probability that out of the first 10 texts, you completely ignore 3, read but don't respond to 2, and respond to 5?

Solution:

$$\frac{10!}{3!2!5!}(1/6)^3(1/3)^2(1/2)^5$$

- (e) Number the next text message 1, the next one after that 2, and so on. Let X be the number of the first text message that you completely ignore. Find $P(X < 5)$.

Solution: The event $X < 5$ is equivalent to the event that you completely ignore at least one of the first four text messages. This is the complement of the event you don't completely ignore any of the first four text messages. This event has probability $(5/6)^4$, so

$$P(X < 5) = 1 - (5/6)^4.$$

[16 points] 2. Let X and Y have joint density

$$f(x, y) = \begin{cases} x + y & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Find the marginal probability density function of X . **Fully compute all integrals. Be sure to clearly state where the pdf is zero and where it's nonzero.**

Solution: For $0 < x < 1$,

$$f_X(x) = \int_0^1 (x + y) dy = \left(xy + \frac{y^2}{2} \right) \Big|_{y=0}^{y=1} = x + \frac{1}{2}.$$

Outside of this interval, the density is equal to zero.

- (b) Find $E(e^{XY})$. **Set up integrals but do not compute them.**

Solution:

$$E(e^{XY}) = \int_0^1 \int_0^1 e^{xy} (x + y) dx dy.$$

- (c) Find $P(Y < X)$. **Set up integrals but do not compute them.**

Solution:

$$P(Y < X) = \int_0^1 \int_0^x (x + y) dy dx.$$

- (d) Find $P(Y < X^2)$. **Set up integrals but do not compute them.**

Solution:

$$P(Y < X^2) = \int_0^1 \int_0^{x^2} (x + y) dy dx.$$

- [8 points] 3. Let $X \sim \text{Unif}[0, 1]$, and let $Y = X^3$. Compute the probability density function of Y .
Be sure to clearly state where the pdf is zero and where it's nonzero.

Solution: First, we compute the cumulative distribution function of Y . For $0 \leq y \leq 1$,

$$P(Y \leq y) = P(X^3 \leq y) = P(X \leq y^{1/3}) = y^{1/3}.$$

So, the cumulative distribution function of Y is

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0, \\ y^{1/3} & \text{if } 0 \leq y \leq 1 \\ 1 & \text{if } y > 1. \end{cases}$$

Now, we take the derivative to find the density of Y :

$$f_Y(y) = \begin{cases} \frac{1}{3}y^{-2/3} & \text{if } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

[12 points] 4. There are 9 women and 6 men in a room. **As usual, there's no need to compute numerical final answers in this problem.**

- (a) A person is chosen uniformly at random. What's the probability it's a woman?

Solution: $\frac{9}{15}$.

- (b) You and I independently choose a person at random from the room. What's the probability that at least one of the people we choose is a woman?

Solution: The probability that both are men is $(6/15)^2$. This is the complement of the event that at least one is a woman, which therefore has probability $1 - (6/15)^2$.

- (c) We choose two different people from the room, uniformly at random. What's the probability that both are women?

Solution: We can think of selecting the group of two as choosing a person uniformly at random, and then choosing a second person uniformly at random from those remaining. Let A be the event that the first person chosen is a woman, and B the event that the second person is. Then $P(A) = 9/15$. Given event A , there are 14 people remaining, 8 of whom are women. So, $P(B | A) = 8/14$. Hence

$$P(AB) = P(A)P(B | A) = \frac{9}{15} \cdot \frac{8}{14}.$$

A different approach is to ignore order. Then there are $\binom{15}{2}$ equally likely outcomes for who is chosen. The number of outcomes in which two women are chosen is $\binom{9}{2}$. Hence the probability of choosing two women is

$$\frac{\binom{9}{2}}{\binom{15}{2}}.$$

- [12 points] 5. Let X and Y be discrete random variables with joint probability mass function given by the following table:

		Y		
		1	2	3
X	1	.1	.2	0
	2	.3	.1	.1
	3	0	.1	.1

- (a) Give the marginal probability mass functions of X and Y .

Solution:

$$\begin{aligned} p_X(1) &= .3, & p_X(2) &= .5, & p_X(3) &= .2 \\ p_Y(1) &= .4, & p_Y(2) &= .4, & p_Y(3) &= .2. \end{aligned}$$

- (b) Are X and Y independent? Justify your answer. (No credit will be given without justification.)

Solution: No. For example, $P(X = 3, Y = 1) = 0$, but $P(X = 3)P(Y = 1) = (.2)(.4) = .08$.

- (c) What is $P(X = 2 \mid Y = 3)$?

Solution:

$$P(X = 2 \mid Y = 3) = \frac{P(X = 2, Y = 3)}{P(Y = 3)} = \frac{.1}{.2} = \frac{1}{2}.$$

[12 points] 6. A company's revenue on different days form a collection of independent and identically distributed random variables with mean \$1000 and variance 900.

- (a) Let X be the company's revenue tomorrow. From the information given to you, give the best available bound on the probability that X is bigger than \$1060.

Solution: By Chebyshev's inequality,

$$P(X > 1060) = P(X - 1000 > 60) \leq P(|X - 1000| > 60) \leq \frac{900}{60^2} = \frac{1}{4}.$$

- (b) Let Y be the company's revenue over the next 10 days. Use the central limit theorem to give an estimate on the probability that Y is smaller than \$9900. Write your answer in terms of $\Phi(x)$, the cumulative distribution function of the standard normal distribution.

Solution: Let $X_1, \dots, X_{10}$ be the company's revenue over each of the next 10 days. Then $Y = X_1 + \dots + X_{10}$. According to the CLT,

$$\begin{aligned} P(Y < 9900) &= P\left(\frac{Y - 10(1000)}{30\sqrt{10}} < \frac{-10}{3\sqrt{10}}\right) \\ &\approx \Phi(-10/3\sqrt{10}). \end{aligned}$$

- [8 points] 7. A company has 5 projects to complete and 7 employees. Each employee is assigned to one of the 5 projects, uniformly at random and independently of each other.

Let X be the number of projects to which somebody has been assigned. Find $E(X)$.

Solution: Let J_i be an indicator on someone being assigned project i , so that $X = J_1 + \cdots + J_5$. We compute

$$E(J_i) = P(\text{project } i \text{ has someone assigned to it}) = 1 - (4/5)^7.$$

So,

$$E(X) = E(J_1) + \cdots + E(J_5) = 5\left(1 - (4/5)^7\right).$$

[12 points] 8. Let $X \sim \text{Exp}(2)$. That is, X has density

$$f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let Y have density

$$f_Y(y) = \begin{cases} 2y & \text{if } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that X and Y are independent.

(a) Find $E(Y)$.

Solution:

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^1 2y^2 dy = \frac{2}{3}.$$

(b) What is the joint density of X and Y ? **Make sure you clearly state where the joint density is equal to zero.**

Solution: The joint density is

$$f(x, y) = \begin{cases} 4e^{-2x}y & \text{if } x > 0 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Find $E(XY)$.

Solution: Since $X \sim \text{Exp}(2)$, we have $E(X) = 1/2$. By independence,

$$E(XY) = E(X)E(Y) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}.$$

You can also do a double integral to find this, but it's a lot more work.