You can answer any FOUR of these problems. If you submit five solutions, I’ll grade only the first four.

Lebesgue measure is denoted by $m$.

You can use any result from chapters 1 and 2 of the textbook, but not homework exercises.

**Problem 1.** Let $(X, \mathcal{M})$ be a measurable space. Show that the characteristic function $\chi_E: X \to \mathbb{R}$ is measurable if and only if $E \in \mathcal{M}$.

**Problem 2.** Let $\{f_n\}$ be a sequence of measurable functions on measure space $(X, \mathcal{M}, \mu)$ and suppose that $f_n \to f$ pointwise. Suppose that $\int |f_n| \, d\mu \leq 1$ for all $n$.

(a) Show that $\int |f| \, d\mu \leq 1$.

(b) Give an example of such a sequence $\{f_n\}$ on $(\mathbb{R}, \mathcal{B}_\mathbb{R}, m)$ where $f_n \not\to f$ in $L^1$.

(c) Suppose that $|f_n| \leq |f|$ for all $n$. Show that $f_n \to f$ in $L^1$.

**Problem 3.** Let $\mu$ be a measure on $X$. Let $E_1, E_2, \ldots$ be measurable sets such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Let

$$ F = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n. $$

Prove that $\mu(F) = 0$.

**Problem 4.** Let $f: \mathbb{R} \to [0, \infty]$ be a Borel measurable function, and for $t \in \mathbb{R}$ define $f_t: \mathbb{R} \to [0, \infty]$ by $f_t(x) = f(t + x)$. (You do not need to show that $f_t$ is measurable.) Show that for any $t$,

$$ \int f \, dm = \int f_t \, dm, $$

where $m$ is Lebesgue measure on $\mathbb{R}$.

**Problem 5.** Let $m^*$ denote Lebesgue outer measure on $\mathbb{R}$, and let $E$ be a subset of $\mathbb{R}$, not necessarily Lebesgue measurable, with $m^*(E) < \infty$. Let $E_n = E \cap [-n, n]$. Prove that $m^*(E_n) \to m^*(E)$ as $n \to \infty$. 

---

Date: March 19, 2024.