Problem 1. Define \( f: \mathbb{R}^n \to \mathbb{R} \) by
\[
f(x) = \frac{1}{m(B_1(0))} \chi_{B_1(0)}(x).
\]
Show that the Hardy-Littlewood maximal function \( Hf \) is not integrable. \textit{Hint:} Show that \( Hf(x) \geq c|x|^{-n} \) for some \( c \) and all large enough \( |x| \). You can use Corollary 2.52 even though we didn’t officially cover it.

Commentary: this problem shows that we can have \( f \in L^1 \) but \( Hf \notin L^1 \).

Problem 2. (Folland 3.23) A useful variant of the Hardy-Littlewood maximal function is
\[
H^*f(x) = \sup \left\{ \frac{1}{m(B)} \int_B |f(y)| \, dy : B \text{ is an open ball containing } x \right\}.
\]
Show that \( Hf \leq H^*f \leq 2^n Hf \) for any \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \).

Problem 3. (Folland 3.24) Show for any \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) that if \( f \) is continuous at \( x \), then \( x \) is in the Lebesgue set of \( f \).

Problem 4. (Folland 3.25) If \( E \) is a Borel set in \( \mathbb{R}^n \), the \textit{density} \( D_E(x) \) of \( E \) at \( x \) is defined as
\[
D_E(x) = \lim_{r \to 0} \frac{m(E \cap B_r(x))}{m(B_r(x))}
\]
whenever the limit exists.

(a) Show that \( D_E(x) = 1 \) for a.e. \( x \in E \) and \( D_E(x) = 0 \) for a.e. \( x \in E^c \).
(b) Find examples of \( E \) and \( x \) such that \( D_E(x) \) is a given \( \alpha \in (0, 1) \) and such that \( D_E(x) \) doesn’t exist.

\textit{Note:} This may or may not be helpful—there is more than one way to construct examples in part (b)—but feel free to compute integrals on \( \mathbb{R}^2 \) using polar coordinates (Theorem 2.49 in Folland, though that’s in greater generality).