PROBLEM SET
MTH 70200
REAL ANALYSIS

Problem 1. Define $f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \frac{1}{m(B_1(0))}\chi_{B_1(0)}(x).$$

Show that the Hardy-Littlewood maximal function Hf is not integrable. *Hint:* Show that $Hf(x) \geq c|x|^{-n}$ for some c and all large enough $|x|$. You can use Corollary 2.52 even though we didn't officially cover it.
Commentary: this problem shows that we can have $f \in L^1$ but $Hf \notin L^1$.

Problem 2. (Folland 3.23) A useful variant of the Hardy-Littlewood maximal function is

$$H^*f(x) = \sup\left\{\frac{1}{m(B)}\int_B |f(y)|\,dy \colon B \text{ is an open ball containing } x\right\}.$$

Show that $Hf \leq H^*f \leq 2^n Hf$ for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Problem 3. (Folland 3.24) Show for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ that if f is continuous at x, then x is in the Lebesgue set of f.

Problem 4. (Folland 3.25) If E is a Borel set in $\mathbb{R}^n$, the *density* $D_E(x)$ of E at x is defined as

$$D_E(x) = \lim_{r\to 0}\frac{m\big(E \cap B_r(x)\big)}{m\big(B_r(x)\big)}$$

whenever the limit exists.

(a) Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.
(b) Find examples of E and x such that $D_E(x)$ is a given $\alpha \in (0,1)$ and such that $D_E(x)$ doesn't exist.

Note: This may or may not be helpful—there is more than one way to construct examples in part (b)—but feel free to compute integrals on $\mathbb{R}^2$ using polar coordinates (Theorem 2.49 in Folland, though that's in greater generality).

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