Problem 1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative Borel-measurable function, and let $\varphi(t) = m(\{x: f(x) > t\})$, where $m$ denotes Lebesgue measure. Show that $\varphi$ is right-continuous and decreasing and that

$$\int_0^\infty \varphi(t) \, dt = \int f(x) \, dx.$$ 

Problem 2. Let $f \in L^1(\mathbb{R}, dm)$, where $m$ denotes Lebesgue measure. Adopt the notation $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $A = \{x \in \mathbb{R}^n: x_1 < \cdots < x_n\}$. With $m^n$ denoting $n$-dimensional Lebesgue measure, prove that

$$\int_A f(x_1) \cdots f(x_n) \, dm^n(x) = \frac{1}{n!} \left( \int f(x) \, dx \right)^n.$$ 

Hint: consider how the integral behaves under permutation of the coordinates of $x$.

Problem 3. Let $q_1, q_2, \ldots$ be an enumeration of the rational numbers in $[0, 1]$. Prove that

$$\sum_{n=1}^\infty (-1)^n n^{-3/2} |x - q_n|^{1/2 - q_n}$$

converges to a finite limit for Lebesgue–almost every $x \in [0, 1]$.

Problem 4. (Folland 3.4) Suppose that $\nu$ is a signed measure and $\lambda, \mu$ are positive measures such that $\nu = \lambda - \mu$. Prove that $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

Problem 5. (Folland 3.5) Suppose that $\nu_1$ and $\nu_2$ are finite signed measures. Prove that $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. Hint: Use the previous problem and you should get a very short solution.

Problem 6. Show that the distance function $d(\mu, \nu) = |\nu - \mu|(X)$ is a metric on the vector space of finite signed measures on $(X, \mathcal{M})$.