PROBLEM SET
MTH 70200
REAL ANALYSIS

Problem 1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative Borel-measurable function, and let $\varphi(t) = m(\{x\colon f(x) > t\})$, where m denotes Lebesgue measure. Show that φ is right-continuous and decreasing and that

$$\int_0^\infty \varphi(t)\,dt = \int f(x)\,dx.$$

Problem 2. Let $f \in L^1(\mathbb{R}, dm)$, where m denotes Lebesgue measure. Adopt the notation $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Let

$$A = \{x \in \mathbb{R}^n\colon x_1 < \cdots < x_n\}.$$

With m^n denoting n-dimensional Lebesgue measure, prove that

$$\int_A f(x_1)\cdots f(x_n)\,dm^n(x) = \frac{1}{n!}\left(\int f(x)\,dx\right)^n.$$

Hint: consider how the integral behaves under permutation of the coordinates of x.

Problem 3. Let $q_1, q_2, \dots$ be an enumeration of the rational numbers in $[0,1]$. Prove that

$$\sum_{n=1}^{\infty}(-1)^n n^{-3/2}|x - q_n|^{\frac{1}{2}-q_n}$$

converges to a finite limit for Lebesgue–almost every $x \in [0,1]$.

Problem 4. (Folland 3.4) Suppose that ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$. Prove that $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

Problem 5. (Folland 3.5) Suppose that ν_1 and ν_2 are finite signed measures. Prove that $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. *Hint: Use the previous problem and you should get a very short solution.*

Problem 6. Show that the distance function $d(\mu, \nu) = |\nu - \mu|(X)$ is a metric on the vector space of finite signed measures on $(X, \mathcal{M})$.

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