PROBLEM SET II
MTH 70200
REAL ANALYSIS

Problem 1. Consider the following increasing, right-continuous function:

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

a) Describe the associated Borel measure μ_F.
b) Let μ_0 be the premeasure defined on the algebra of disjoint unions of h-intervals induced by F, and let μ^* be the outer measure induced by μ_0. (This makes μ_F the restriction of μ^* to the Borel σ-algebra.) We mentioned in class that the domain of μ_F could also be taken to be the larger class of μ^*-measurable sets. Which sets are μ^*-measurable?

Problem 2. (Folland 1.28) Let F be increasing and right-continuous, and let μ_F be the associated measure. Show that $\mu_F(\{a\}) = F(a) - F(a-)$, $\mu_F([a,b)) = F(b-) - F(a-)$, $\mu_F([a,b]) = F(b) - F(a-)$, and $\mu_F((a,b)) = F(b-) - F(a)$. (Here $F(x-)$ denotes $\lim_{y \uparrow x} F(y)$.)

Commentary on Problem 2: a point x such that $\mu(\{x\}) > 0$ is called an *atom* of μ. Thus μ_F has no atoms if and only if F is continuous.

Problem 3. For a set $E \subset \mathbb{R}$, let $\operatorname{diam}(E) = \sup E - \inf E$. Let μ be a **finite** Borel measure on $\mathbb{R}$ with no atoms. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\operatorname{diam}(E) < \delta \implies \mu(E) < \epsilon.$$

Problem 4. Let m^* be the outer measure on $\mathbb{R}$ induced by the premeasure on h-intervals associated with some increasing, right-continuous $F\colon \mathbb{R} \to \mathbb{R}$. Suppose that $A \subseteq \mathbb{R}$ has the property that for any real $a < b$,

$$m^*\big(A \cap (a,b]\big) \leq \tfrac{1}{2} m^*\big((a,b]\big).$$

Prove that $m^*(A) = 0$. (*Hint:* First assume that A is bounded.)

Problem 5. (Quals Spring 2016) Let $(X, \mathcal{M}_1, \mu_1)$ and $(X, \mathcal{M}_2, \mu_2)$ be measure spaces on the same set X. Let μ_1^* and μ_2^* be the outer measures on X generated by μ_1 and μ_2, respectively. Prove that $\mu_1^* = \mu_2^*$ if and only if $\mu_2^*|_{\mathcal{M}_1} = \mu_1$ and $\mu_1^*|_{\mathcal{M}_2} = \mu_2$.

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