PROBLEM SET
MTH 70200
REAL ANALYSIS

Problem 1. (Folland 3.37) Suppose $F\colon \mathbb{R} \to \mathbb{C}$. Show that there is a constant M such that $|F(x) - F(y)| \leq M|x - y|$ (that is, F is *Lipschitz continuous*) if and only if F is absolutely continuous and $|F'| \leq M$ a.e.

Problem 2. (Folland 3.39) Suppose that $\{F_j\}$ is a sequence of nonnegative increasing functions on $[a, b]$ and that $F(x) = \sum_{j=1}^{\infty} F_j(x) < \infty$ for all $x \in [a, b]$. Prove that $F'(x) = \sum_{j=1}^{\infty} F_j'(x)$ for a.e. $x \in [a, b]$. *Hint:* It suffices to assume that $F_j \in NBV$. Consider the measures μ_{F_j}.

Problem 3. (Folland 3.40) Let F denote the Cantor function on $[0, 1]$, and set $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$. Let $\{[a_n, b_n]\}$ be an enumeration of the closed subintervals of $[0, 1]$ with rational endpoints, and let

$$F_n(x) = F\left(\frac{x - a_n}{b_n - a_n}\right).$$

Then $G = \sum_{n=1}^{\infty} 2^{-n} F_n$ is continuous and strictly increasing on $[0, 1]$, and $G' = 0$ a.e. *Hint:* use the previous exercise.

Problem 4. (Folland 6.7) Suppose that $f \in L^p \cap L^\infty$ for some $p < \infty$, so that $f \in L^q$ for $q > p$. Show that $\|f\|_\infty = \lim_{q \to \infty} \|f\|_q$.

Problem 5. (Folland 6.13) Recall that a metric space is called *separable* if it has a countable dense subset. Show that $L^p(\mathbb{R}^n, m)$ is separable for $1 \leq p < \infty$, but $L^\infty(\mathbb{R}^n, m)$ is not separable. *Hint:* Find an uncountable collection of functions in L^∞ that all have L^∞-distance at least 1 from each other.

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