PROBLEM SET
MTH 70200
REAL ANALYSIS

Problem 1. Let μ be a finite signed Borel measure on $\mathbb{R}$ and let $F(x) = \mu\big((-\infty, x]\big)$. Show that $|\mu|(\mathbb{R})$ is equal to $T_F(\infty)$, the total variation of F on $\mathbb{R}$.

Hint: Show that $|\mu|(\mathbb{R}) \geq T_F(\infty)$ and $|\mu|(\mathbb{R}) \leq T_F(\infty)$. The hard direction is $|\mu|(\mathbb{R}) \leq T_F(\infty)$. Let $\mathbb{R} = P \cup N$ be a Hahn decomposition for μ. Approximate P by a finite disjoint union A of h-intervals so that $|\mu|(A \bigtriangleup P) = |\mu|(A^c \bigtriangleup N) < \epsilon$ (you can use Proposition 1.20 together with continuity from below).

Problem 2. Let μ be a finite signed Borel measure on $\mathbb{R}$ and let $F(x) = \mu\big((-\infty, x]\big)$. Let $F = F^+ - F^-$ be the Jordan decomposition of F. Show that $\mu^+ = \mu_{F^+}$ and $\mu^- = \mu_{F^-}$, i.e., the Jordan decomposition of μ corresponds to the Jordan decomposition of F.

Hint: Apply the previous exercise to the restriction of μ to $(-\infty, x]$ and use Folland Exercise 3.4. Your solution should be very short if you do it right.

Problem 3. (Folland 3.30) Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is $\mathbb{Q}$.

Problem 4. (Folland 3.33) Let $F\colon \mathbb{R} \to \mathbb{R}$ be increasing. Prove that $F(b) - F(a) \geq \int_a^b F'(t)\, ft$.

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