PROBLEM SET
MTH 70200
REAL ANALYSIS

Problem 1. Let $\mu$ be a finite signed Borel measure on $\mathbb{R}$ and let $F(x) = \mu((-\infty, x])$. Show that $|\mu|(\mathbb{R})$ is equal to $T_F(\infty)$, the total variation of $F$ on $\mathbb{R}$.

Hint: Show that $|\mu|(\mathbb{R}) \geq T_F(\infty)$ and $|\mu|(\mathbb{R}) \leq T_F(\infty)$. The hard direction is $|\mu|(\mathbb{R}) \leq T_F(\infty)$. Let $\mathbb{R} = P \cup N$ be a Hahn decomposition for $\mu$. Approximate $P$ by a finite disjoint union $A$ of h-intervals so that $|\mu|(A \triangle P) = |\mu|(A' \triangle N) < \epsilon$ (you can use Proposition 1.20 together with continuity from below).

Problem 2. Let $\mu$ be a finite signed Borel measure on $\mathbb{R}$ and let $F(x) = \mu((-\infty, x])$. Let $F = F^+ - F^-$ be the Jordan decomposition of $F$. Show that $\mu^+ = \mu_{F^+}$ and $\mu^- = \mu_{F^-}$, i.e., the Jordan decomposition of $\mu$ corresponds to the Jordan decomposition of $F$.

Hint: Apply the previous exercise to the restriction of $\mu$ to $(-\infty, x]$ and use Folland Exercise 3.4. Your solution should be very short if you do it right.

Problem 3. (Folland 3.30) Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is $\mathbb{Q}$.

Problem 4. (Folland 3.33) Let $F: \mathbb{R} \to \mathbb{R}$ be increasing. Prove that $F(b) - F(a) \geq \int_a^b F'(t) ft$. 

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