Math 311, Midterm 2
November 26, 2025

Name: ______________________________

I pledge that I have neither given nor received unauthorized assistance during this examination. **Signature:**

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- No calculators or electronic devices are allowed.
- Unless the problem specifically says, you **do not** need to calculate the final answer. An answer that looks something like

$$\frac{\binom{6}{2}\binom{8}{5}}{\binom{23}{4}} \quad \text{or} \quad \frac{(12)(11)(10)+(11)(10)(9)}{14!}$$

  is entirely acceptable.
- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 8 problems on 10 pages.

Question	Points	Score
1	18	
2	12	
3	9	
4	6	
5	11	
6	6	
7	10	
8	12	
Total:	84	

Good luck!

1. A company manufactures screws. Each screw they make is defective with probability 1/50.

[2 points] (a) Let X be the number of the first defective screw. (For example, if the first three screws are good and the fourth is defective, then $X = 4$.) What is $E(X)$?

> **Solution:** Since $X \sim \text{Geo}(1/50)$, using the formula for expectation of the geometric distribution, we have $E(X) = 50$.

[3 points] (b) With X as in the last problem, what is $P(X > 25)$?

> **Solution:** The event $\{X > 25\}$ occurs when the first 25 screws are good, which happens with probability
>
> $$\left(\frac{49}{50}\right)^{25}.$$

[3 points] (c) With X as in the last problem, what is $P(X = 10)$?

> **Solution:** The event $\{X = 10\}$ occurs when the first nine screws are good and the next is defective, which occurs with probability
>
> $$\left(\frac{49}{50}\right)^{9}\left(\frac{1}{50}\right).$$

[2 points] (d) What is the distribution of X? (Your answer should give the name and parameter of the distribution.)

Solution: It's Geo(1/50).

[2 points] (e) Let Y be the number of defective screws in the first 200. What is $E(Y)$?

Solution: Since $Y \sim \text{Bin}(200, 1/50)$, using the formula for expectation of the binomial distribution we have

$$E(Y) = (200)\tfrac{1}{50} = 4.$$

[3 points] (f) With Y as in the last problem, what is $P(Y \leq 2)$?

Solution:

$$P(Y \leq 2) = \left(\frac{49}{50}\right)^{200} + \binom{200}{1}\left(\frac{49}{50}\right)^{199}\left(\frac{1}{50}\right) + \binom{200}{2}\left(\frac{49}{50}\right)^{198}\left(\frac{1}{50}\right)^{2}$$

[3 points] (g) With Y as in the last problem, what is $P(Y = 1 \mid Y \geq 1)$?

Solution: We have to use the definition of conditional probability directly here:

$$\begin{aligned} P(Y = 1 \mid Y \geq 1) &= \frac{P(Y=1)}{P(Y \geq 1)} \\ &= \frac{\binom{200}{1}\left(\frac{49}{50}\right)^{199}\left(\frac{1}{50}\right)}{1 - \left(\frac{49}{50}\right)^{200}} \approx 7.31\% \end{aligned}$$

[12 points] 2. The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{24}(2x+3y^2) & \text{if } 0 \le x \le 2 \text{ and } 0 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the marginal probability density function of X? **Please make it clear if the density you find is only for an interval.**

Solution:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{24}\int_0^2 (2x+3y^2)\,dy = \frac{1}{24}\big(2xy+y^3\big)\Big|_{y=0}^{y=2}$$
$$= \frac{1}{24}\big(4x+8\big).$$

(b) What is $P(X < 1 \text{ and } Y > 1)$? **Give a final, numerical answer.**

Solution: Compute

$$\frac{1}{24}\int_0^1\int_1^2 \big(2x+3y^2\big)\,dy\,dx = \frac{1}{24}\int_0^1 \big(2xy+y^3\big)\Big|_{y=1}^{y=2}\,dx$$
$$= \frac{1}{24}\int_0^1 \big(2x+7\big)\,dx$$
$$= \frac{1}{24}\big(x^2+7x\big)\Big|_0^1 = \frac{1}{3}.$$

(c) Set up an integral to find $P(Y > X^2)$. **Your answer should be a parametrized, double integral, but don't compute it.**

Solution:

$$\int_0^{\sqrt{2}} \int_{x^2}^{2} \frac{1}{24}\left(2x + 3y^2\right) dy\, dx$$

(d) Set up an integral to compute $E(e^{XY})$. **Your answer should be a parametrized, double integral, but don't compute it.**

Solution:

$$\int_0^{2} \int_0^{2} \frac{1}{24}\left(2x + 3y^2\right) e^{xy}\, dx\, dy$$

[9 points] 3. Let X have probability density function $f(x) = \frac{3}{2}x^2$ on the interval $[-1, 1]$. **In this question, compute a final numerical answer for all questions.**

(a) Find $P(X > \frac{1}{2})$.

Solution:

$$P\left(X > \tfrac{1}{2}\right) = \int_{1/2}^{1} \tfrac{3}{2}x^2\,dx = \tfrac{1}{2}x^3\Big|_{1/2}^{1} = 1/2 - 1/16 = 7/16.$$

(b) Find $E(X)$.

Solution:

$$E(X) = \int_{-1}^{1} x\left(\tfrac{3}{2}\right)x^2\,dx = \tfrac{3}{8}x^4\Big|_{-1}^{1} = \tfrac{3}{8} - \tfrac{3}{8} = 0$$

Or you can just note that the density is symmetric around 0, and therefore the expectation of the random variable is 0.

(c) Find $\mathrm{Var}(X)$.

Solution: First we compute

$$E(X^2) = \int_{-1}^{1} x^2\left(\tfrac{3}{2}\right)x^2\,dx = \tfrac{3}{10}x^5\Big|_{-1}^{1} = \tfrac{3}{10} - \tfrac{3}{10} = .6$$

And the variance is

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = .6 - 0^2 = .6.$$

[6 points] 4. In about 10% of all years, there is a significant earthquake in Turkey. Assuming that each year is independent, use the normal approximation to estimate the probability that over the last 10000 years, there were more than 1060 containing significant earthquakes. **Give your answer in terms of $\Phi(x)$, the cumulative distribution function of the standard normal distribution.**

Solution: Let $S \sim \text{Bin}(10000, .1)$. Then $\mu = E(S) = 10000(.1) = 1000$ and $\sigma^2 = \text{Var}(S) = 10000(.1)(.9) = 900$. We can approximate $P(S > 1060)$ as

$$\begin{aligned} P(S > 1060) &= P(\frac{S-\mu}{\sigma} > \frac{1060-\mu}{\sigma}) \\ &= P\left(\frac{S-1000}{30} > \frac{1060-1000}{30}\right) \\ &\approx P\left(Z > \frac{1060-1000}{30}\right) = P(Z > 2) = 1 - \Phi(2), \end{aligned}$$

where Z has the standard normal distribution.

5. The following table gives the joint mass function of X and Y:

		Y = 1	2	3
	1	.3	.2	0
X	2	.2	.1	0
	3	0	.1	.1

[2 points] (a) Give the marginal probability mass functions of X and Y.

Solution:
$p_X(1) = .5$, $p_X(2) = .3$, $p_X(3) = .2$.
$p_Y(1) = .5$, $p_Y(2) = .4$, $p_Y(3) = .1$.

[3 points] (b) What is $P(X = 3 \mid Y = 2)$?

Solution:

$$P(X = 3 \mid Y = 2) = \frac{P(X = 3, Y = 2)}{P(Y = 2)} = \frac{.1}{.4} = \frac{1}{4}.$$

[3 points] (c) What is $P(X = Y + 1)$?

Solution:

$$P(X = Y + 1) = P(X = 2, Y = 1) + P(X = 3, Y = 2) = .2 + .1 = .3$$

[3 points] (d) What is $E(XY)$?

Solution:

$$\begin{aligned} .3(1)(1) + &.2(1)(2) + .2(2)(1) + .1(2)(2) + .1(3)(2) + .1(3)(3) \\ &= .3 + .4 + .4 + .4 + .6 + .9 = 3 \end{aligned}$$

[6 points] 6. Let $X \sim \text{Unif}[-1,1]$ and let $Y = X^3$. Find the probability density function of Y. **Please make it clear if the density you find is only for an interval.**

Solution: We compute the cdf of Y. For $y \in [-1,1]$,

$$F_Y(y) = P(Y \le y) = P(X^3 \le y) = P(X \le y^{1/3}) = \frac{y^{1/3} - (-1)}{2} = \frac{y^{1/3}+1}{2}.$$

Now

$$f_Y(y) = F'_Y(y) = \tfrac{1}{6} y^{-2/3}.$$

For $y < -1$ or $y > 1$, we have $f_Y(y) = 0$. So the density of Y is $\frac{1}{6}y^{-2/3}$ on the interval $[-1,1]$.

Note that the density is infinite at $x = 0$, which doesn't cause any problem.

7. Let X be the number of successes in 10000 independent trials with success probability 1/20000.

[2 points] (a) What is the **exact** distribution of X? (Your answer should be a named distribution and you should give its parameter(s).)

Solution: $X \sim \text{Bin}(10000, 1/20000)$

[3 points] (b) What is the **exact** value of $P(X \leq 1)$? (You just need to write down an expression, not compute the numerical value.)

Solution:

$$\left(1 - \frac{1}{20000}\right)^{10000} + \binom{10000}{1}\left(1 - \frac{1}{20000}\right)^{9999}\left(\frac{1}{20000}\right)$$

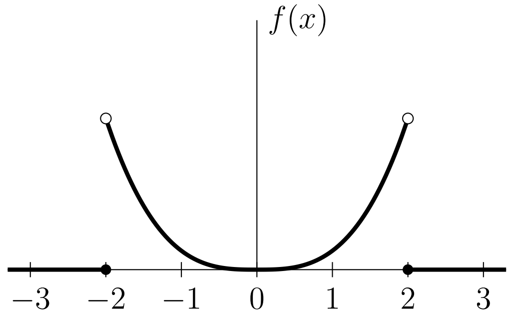
[2 points] (c) What distribution is appropriate to **approximate** the distribution of X? (Your answer should be a named distribution and you should give its parameter(s).)

Solution: $\text{Poi}(1/2)$

[3 points] (d) What is the **approximate** value of $P(X \leq 1)$ according to this distribution?

Solution: $e^{-1/2} + e^{-1/2}(1/2) = \frac{3}{2}e^{-1/2}$
Of course you can't see this without a calculator, but the exact answer and the approximation match to 5 decimal places.

[12 points] 8. Random variable X has probability density function $f(x)$ depicted below:



Let $Y = X^2$. Say whether the following statements about X and Y are true or false. You don't need to explain your answers.

(a) $P(X = 1.5) = 0$ √ **True** ○ False

(b) $P(X > 0) = 1$ ○ True √ **False**

(c) $P(Y > 0) = 1$ √ **True** ○ False

(d) $P(-2 < X < 2) = 1$ √ **True** ○ False

(e) $P(-2 < Y < 2) = 1$ ○ True √ **False**

(f) $P\big(0 \leq X \leq 1\big) < P\big(1 \leq X \leq 2\big)$ √ **True** ○ False