

*I pledge that I have neither given nor received
unauthorized assistance during this examination.*

Signature:

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- No calculators or electronic devices are allowed.
- Unless the problem specifically says, you **do not** need to calculate the final answer. An answer that looks something like

$$\frac{\binom{6}{2}\binom{8}{5}}{\binom{23}{4}} \quad \text{or} \quad \frac{(12)(11)(10) + (11)(10)(9)}{14!}$$

is entirely acceptable.

- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 7 problems on 7 pages.

Question	Points	Score
1	9	
2	6	
3	4	
4	8	
5	6	
6	8	
7	6	
Total:	47	

Good luck!

- [9 points] 1. An urn contains 5 white balls, 6 black balls, and 9 red balls. You reach into the urn, grab three balls, and pull them all out.

(a) What is the probability that all three balls are white?

Solution: Using a sample space for sampling without replacement, order not mattering, the sample space has size $\binom{20}{3}$. The number of outcomes with all balls white is $\binom{5}{3}$. So the probability is

$$\frac{\binom{5}{3}}{\binom{20}{3}} = \frac{1}{114}.$$

Alternatively, you could use the sample space where order does matter, and let W_1 , W_2 , and W_3 be the probabilities of the first, second, and third balls being white. Then

$$\begin{aligned} P(W_1 W_2 W_3) &= P(W_1) P(W_2 \mid W_1) P(W_3 \mid W_1 W_2) \\ &= \frac{5}{20} \cdot \frac{4}{19} \cdot \frac{3}{18} = \frac{1}{114}. \end{aligned}$$

(b) What is the probability that exactly two of three balls are white?

Solution: Use the sample space for sampling without replacement where order doesn't matter. The number of outcomes with two white balls and one nonwhite ball is $\binom{5}{2} \binom{15}{1}$. So the probability is

$$\frac{\binom{5}{2} \binom{15}{1}}{\binom{20}{3}}$$

(c) Conditional on at least two of the balls being white, what is the probability that exactly two are white?

Solution: Let W_2 be the event that exactly two balls are white from the last part. Let W_3 be the event that all three balls are white, from the first part. We're looking for

$$\begin{aligned} P(W_2 \mid W_2 \cup W_3) &= \frac{P(W_2 \cap (W_2 \cup W_3))}{P(W_2 \cup W_3)} = \frac{P(W_2)}{P(W_2 \cup W_3)} \\ &= \frac{\frac{\binom{5}{2} \binom{15}{1}}{\binom{20}{3}}}{\frac{\binom{5}{2} \binom{15}{1}}{\binom{20}{3}} + \frac{\binom{5}{3}}{\binom{20}{3}}} \end{aligned}$$

- [6 points] 2. Coin #1 and coin #2 are fair, but coin #3 lands on tails with probability $3/4$.

You choose one of three coins uniformly at random, and you flip it and get tails. What is the probability that you chose coin #3?

Solution: Let T be the event of getting tails, and C_1 , C_2 , and C_3 the (disjoint) events that you chose coin #1, #2, or #3. We know that

$$P(T \mid C_1) = P(T \mid C_2) = 1/2$$

and

$$P(T \mid C_3) = 3/4.$$

Also we know that $P(C_1) = P(C_2) = P(C_3) = 1/3$. We want to find $P(C_3 \mid T)$. By Bayes' formula, it's

$$\begin{aligned} P(C_3 \mid T) &= \frac{P(C_3 T)}{P(T)} = \frac{P(C_3)P(T \mid C_3)}{P(C_1)P(T \mid C_1) + P(C_2)P(T \mid C_2) + P(C_3)P(T \mid C_3)} \\ &= \frac{\frac{1}{3} \cdot \frac{3}{4}}{\frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{3}{4}} = \frac{\frac{1}{4}}{\frac{1}{6} + \frac{1}{6} + \frac{1}{4}} = \frac{3}{2 + 2 + 3} = \frac{3}{7}. \end{aligned}$$

[4 points] 3. For each collection of events, say whether you would expect them to be **disjoint**, **independent**, or **neither**.

(a) Three coins are flipped.

$$A = \{\text{coins \#1 and \#3 are heads}\}$$

$$B = \{\text{coins \#1, \#2, and \#3 are tails}\}$$

☒ **disjoint** ☐ independent ☐ neither

(b) Three coins are flipped.

$$A = \{\text{coins \#1 and \#3 are heads}\}$$

$$B = \{\text{coins \#1, \#2, and \#3 are heads}\}$$

☐ disjoint ☐ independent ☒ **neither**

(c) Two strangers walk into a room.

$$A = \{\text{person \#1 was born on a Monday}\}$$

$$B = \{\text{person \#2 was born on a Tuesday}\}$$

Events A and B are:

☐ disjoint ☒ **independent** ☐ neither

(d) One person walks into a room.

$$A = \{\text{person was born on a Monday}\}$$

$$B = \{\text{person was born on a Tuesday}\}$$

Events A and B are:

☒ **disjoint** ☐ independent ☐ neither

4. An urn contains balls numbered $1, \dots, 20$. One ball is chosen, then a second, then a third, and then a fourth. Balls are not reinserted into the urn, so each ball can only be selected once.

[2 points]

- (a) How large is the natural sample space for this random experiment?

Solution: $(20)_4 = (20)(19)(18)(17)$.

[3 points]

- (b) What is the probability that exactly two of the four chosen balls have an even number on them?

Solution: Here use the sample space where order doesn't matter. There are 10 even balls and 10 odd ones, so the number of outcomes with two of each is $\binom{10}{2}\binom{10}{2}$, giving us a probability of

$$\frac{\binom{10}{2}\binom{10}{2}}{\binom{20}{4}}.$$

[3 points]

- (c) What is the probability that the first ball is even, the second is odd, the third is even, and the fourth is odd?

Solution: Let E_1 be the event that the first ball is even, O_2 that the second is odd, E_3 that the third is even, and O_4 that the fourth is odd.

$$\begin{aligned} P(E_1 O_2 E_3 O_4) &= P(E_1)P(O_2 \mid E_1)P(E_3 \mid O_2 E_1)P(O_4 \mid E_3 O_2 E_1) \\ &= \frac{10}{20} \frac{10}{19} \frac{9}{18} \frac{9}{17}. \end{aligned}$$

- [6 points] 5. You roll a die five times. If you get exactly three appearances of some number and two of another, call the rolls a “full house”. (For example, it’s a full house if you roll 1, 3, 3, 1, 1.)

What is the probability of rolling a full house?

Solution: First, note that there are 6^5 different elements in the sample space. Now we count the number of full houses.

There are $\binom{5}{2} = 10$ different orders of the numbers a, a, a, b, b , since there are $\binom{5}{2}$ choices for which positions to put the b s. Now, we have 6 choices for what a might be, and 5 choices for what b might be. So in all there are $(10)(6)(5)$ different full house sequences, giving us a probability of

$$\frac{(10)(6)(5)}{6^5} = \frac{25}{648}.$$

6. A random student is picked from a school. There is a 50% chance that the student is wearing a watch, a 40% chance the student is wearing a bracelet, and a 30% chance the student is wearing neither a watch nor a bracelet.

[3 points]

- (a) What is the probability the student is wearing a watch or a bracelet (or both)?

Solution: Let W be the event the student wears a watch and B the event they wear a bracelet. We're told that $P(W) = .5$, $P(B) = .4$, and $P(W^c \cap B^c) = .3$. So,

$$P(W \cup B) = 1 - P((W \cup B)^c) = 1 - P(W^c \cap B^c) = 1 - .3 = .7.$$

[3 points]

- (b) What is the probability the student is wearing a watch and a bracelet?

Solution: By inclusion-exclusion,

$$P(W \cup B) = P(W) + P(B) - P(W \cap B)$$

From the first part, we have

$$.7 = .5 + .4 - P(W \cap B).$$

So, $P(W \cap B) = .2$

[2 points]

- (c) Are the events of wearing a watch and wearing a bracelet independent? Why or why not?

Solution: Yes, they're independent: $.2 = P(W \cap B) = P(W)P(B)$.

- [6 points] 7. There are two kinds of employees at Lab X: technicians and scientists. The lab can only operate if it has at least two technicians and one scientist.

One a certain day, two technicians and two scientists are scheduled to work. Technicians #1 and #2 show up with probability p_1 and p_2 . Scientists #1 and #2 show up with probability q_1 and q_2 . The events of each employee showing up are independent. Find the probability that the lab is able to operate. (Your answer will be in terms of p_1 , p_2 , q_1 , and q_2 .)

Solution: The probability that both technicians show up is p_1p_2 , by independence. The probability that at least one scientist shows up is $q_1+q_2-q_1q_2$, by the inclusion-exclusion formula and independence. The probability that both of these events happen is

$$p_1p_2(q_1 + q_2 - q_1q_2).$$