

*I pledge that I have neither given nor received  
unauthorized assistance during this examination.*

**Signature:**

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- No calculators or electronic devices are allowed.
- Unless the problem specifically says, you **do not** need to calculate the final answer. An answer that looks something like

$$\frac{\binom{6}{2}\binom{8}{5}}{\binom{23}{4}} \quad \text{or} \quad \frac{(12)(11)(10) + (11)(10)(9)}{14!}$$

is entirely acceptable.

- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 7 problems on 10 pages.

| Question | Points | Score |
|----------|--------|-------|
| 1        | 15     |       |
| 2        | 4      |       |
| 3        | 8      |       |
| 4        | 6      |       |
| 5        | 8      |       |
| 6        | 9      |       |
| 7        | 4      |       |
| Total:   | 54     |       |

**Good luck!**

[15 points] 1. There are 18 professors in the mathematics department. Three **different** professors are chosen at random and assigned to the following three jobs:

- chair
- advisor
- scheduler

- (a) What is the probability that the chair is Prof. Rosen, the advisor is Prof. Fuchs, and the scheduler is Prof. Conidis?

**Solution:** We think of the chair, advisor, and scheduler as 3 professors chosen from 18 without replacement, where order matters (the first is the chair, the second the advisor, and the third the scheduler). So, the sample space has  $(18)_3 = (18)(17)(16)$  equally likely outcomes. The event in question is a single outcome, so its probability is

$$\frac{1}{(18)(17)(16)}.$$

- (b) Of the 18 professors, their ranks are as follows: 4 assistant professors, 5 associate professors, and 9 full professors. What is the probability that none of the positions (chair, advisor, scheduler) are held by full professors?

**Solution:** We use the same sample space as in part (a), with  $(18)(17)(16)$  outcomes. There are  $4+5 = 9$  professors who are not full professors. The number of outcomes in which all three jobs go to these 9 professors is  $(9)_3 = (9)(8)(7)$ : there are 9 choices for chair, then 8 for advisor, then 7 for scheduler. So, the probability is

$$\frac{(9)(8)(7)}{(18)(17)(16)}.$$

Alternatively, we could switch to the sample space where order doesn't matter, with  $\binom{18}{3}$  outcomes. The number of outcomes in which only assistant and associate professors are chosen is then  $\binom{9}{3}$ , and the probability is

$$\frac{\binom{9}{3}}{\binom{18}{3}},$$

which is of course equal to the first answer.

- (c) What is the probability that a full professor is chosen as chair, an associate professor is chosen as advisor, and an assistant professor is chosen as scheduler?

**Solution:** The chair can be any of the 9 full professors, the advisor any of the 5 associate professors, and the scheduler any of the 4 assistant professors. So, there are  $(9)(5)(4)$  outcomes in this event, and the probability is

$$\frac{(9)(5)(4)}{(18)(17)(16)}.$$

- (d) What is the probability that the three professors chosen for the jobs are all of different ranks?

**Solution:** If the three professors are all of different ranks, then the ranks of the chair, advisor, and scheduler are some ordering of assistant, associate, and full. There are  $3! = 6$  such orderings, and each one is an event like the one in part (c). For each ordering, the number of outcomes is the same product  $(9)(5)(4)$ , just with the factors in a different order. These six events are disjoint, so the probability is

$$\frac{6(9)(5)(4)}{(18)(17)(16)}.$$

Alternatively, switch to the sample space where order doesn't matter, since we don't care which job each of the three chosen professors has. Then there are  $(9)(5)(4)$  ways to choose three professors of different ranks, and the probability is

$$\frac{(9)(5)(4)}{\binom{18}{3}}.$$

- (e) What is the probability that the chair and advisor have the same rank and the scheduler has a lower rank? (From lowest to highest, the ranks are assistant, associate, full.)

**Solution:** We split into disjoint cases according to the rank shared by the chair and advisor.

- *Chair and advisor are both full professors.* Then the scheduler is an assistant or associate professor. There are 9 choices for chair, then 8 for advisor, then  $4 + 5 = 9$  for scheduler, giving  $(9)(8)(9)$  outcomes.
- *Chair and advisor are both associate professors.* Then the scheduler is an assistant professor. There are 5 choices for chair, then 4 for advisor, then 4 for scheduler, giving  $(5)(4)(4)$  outcomes.

- *Chair and advisor are both assistant professors.* This can't happen, since there is no rank lower than assistant for the scheduler.

Adding up the cases, the probability is

$$\frac{(9)(8)(9) + (5)(4)(4)}{(18)(17)(16)}.$$

(This simplifies to  $\frac{728}{4896} = \frac{91}{612}$ , but you don't need to do this.)

- [4 points] 2. You roll a fair 6-sided die 3 times. Let  $\Omega$  be the set of all triplets  $(a, b, c)$  where  $a$ ,  $b$ , and  $c$  are integers from 1 to 6, i.e.,

$$\Omega = \{(a, b, c) : a, b, c \in \{1, \dots, 6\}\}.$$

Each element in  $\Omega$  represents a sequence of three rolls, as usual. For example,  $(1, 5, 1)$  represents rolling 1, then rolling 5, then rolling 1.

- (a) Let  $X$  denote the sum of the three rolls. Then  $X$  is a/an \_\_\_\_\_.

- ☐ sample space
- ☐ outcome
- ☐ event
- ☒ **random variable**
- ☐ probability measure

- (b) The set  $\{(1, 5, 1), (3, 2, 1)\}$  is a/an \_\_\_\_\_.

- ☐ sample space
- ☐ outcome
- ☒ **event**
- ☐ random variable
- ☐ probability measure

- (c) The element  $(1, 5, 1)$  is a/an \_\_\_\_\_.

- ☐ sample space
- ☒ **outcome**
- ☐ event
- ☐ random variable
- ☐ probability measure

- (d) The set  $\Omega$  is the \_\_\_\_\_ for this random experiment.

- ☒ **sample space**
- ☐ outcome
- ☐ event
- ☐ random variable
- ☐ probability measure

- [8 points] 3. Assume  $1/4$  of all twin births are identical (and hence always have the same sex), while  $3/4$  are fraternal (and hence have the same sex 50% of the time). You learn that Miranda is expecting twins. (*Note:* Assume babies are equally likely to be a boy or a girl.)

(a) What is the probability that Miranda will have two girls?

**Solution:** Let  $I$  be the event that Miranda's twins are identical, so that  $I^c$  is the event that they are fraternal, and let  $G$  be the event that she has two girls. We are given  $P(I) = \frac{1}{4}$  and  $P(I^c) = \frac{3}{4}$ . Identical twins are both girls half the time, so  $P(G | I) = \frac{1}{2}$ . Fraternal twins are the same sex half the time, and when they are, they are both girls half the time, so  $P(G | I^c) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$ .

By the law of total probability,

$$\begin{aligned} P(G) &= P(G | I)P(I) + P(G | I^c)P(I^c) \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = \frac{1}{8} + \frac{3}{16} = \frac{5}{16}. \end{aligned}$$

- (b) You learn that Miranda gave birth to two girls. What is the probability that they are identical?

**Solution:** We want  $P(I | G)$ . Using Bayes' formula, with the numbers from part (a),

$$P(I | G) = \frac{P(I \cap G)}{P(G)} = \frac{P(G | I)P(I)}{P(G)} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)}{\frac{5}{16}} = \frac{\frac{1}{8}}{\frac{5}{16}} = \frac{2}{5}.$$

- [6 points] 4. Rosa's Bakery can open in the morning only if the head baker shows up and at least two of the bakery's three assistants show up. The head baker shows up with probability  $b$ , and each assistant shows up with probability  $a$ . The events of each employee showing up are independent. Find the probability that the bakery is able to open. (Your answer will be in terms of  $b$  and  $a$ .)

**Solution:** Let  $H$  be the event that the head baker shows up, and let  $A_1$ ,  $A_2$ , and  $A_3$  be the events that assistants #1, #2, and #3 show up. We are given  $P(H) = b$  and  $P(A_1) = P(A_2) = P(A_3) = a$ , and that  $H$ ,  $A_1$ ,  $A_2$ ,  $A_3$  are independent.

Let  $E$  be the event that at least two assistants show up. Then the bakery opens on the event  $HE$ . Since  $E$  depends only on the assistants, and the assistants showing up is independent of the head baker showing up,

$$P(HE) = P(H)P(E) = bP(E).$$

To find  $P(E)$ , we write  $E$  in terms of  $A_1$ ,  $A_2$ , and  $A_3$ . At least two assistants show up if either all three show up or exactly two show up, and there are three ways for exactly two to show up, depending on which assistant is missing. So,

$$E = A_1A_2A_3 \cup A_1A_2A_3^c \cup A_1A_2^cA_3 \cup A_1^cA_2A_3.$$

These four events are disjoint, since each one specifies exactly which assistants show up. So,  $P(E)$  is the sum of their probabilities. To compute each of these, we use the fact that if events are independent, then so are their complements (for example,  $A_1$ ,  $A_2$ , and  $A_3^c$  are independent), and that  $P(A_i^c) = 1 - P(A_i) = 1 - a$ . This gives

$$\begin{aligned} P(E) &= P(A_1A_2A_3) + P(A_1A_2A_3^c) + P(A_1A_2^cA_3) + P(A_1^cA_2A_3) \\ &= P(A_1)P(A_2)P(A_3) + P(A_1)P(A_2)P(A_3^c) \\ &\quad + P(A_1)P(A_2^c)P(A_3) + P(A_1^c)P(A_2)P(A_3) \\ &= a^3 + a^2(1-a) + a^2(1-a) + a^2(1-a) \\ &= a^3 + 3a^2(1-a). \end{aligned}$$

Therefore, the probability that the bakery opens is

$$P(HE) = b(a^3 + 3a^2(1-a)).$$

(If you multiply this out, it simplifies to  $b(3a^2 - 2a^3)$ , but you don't need to do this.)

[8 points] 5. Roll a die six times. In the following questions, answer whether you would expect the given two events to be **disjoint**, **independent**, or **neither**.

(a)

$A = \{\text{all rolls are 1}\}$

$B = \{\text{the rolls include a 1, 2, 3, 4, 5, and 6 (in any order)}\}$

☒ **disjoint**    ☐ independent    ☐ neither

**Solution:** If all the rolls are 1, then the rolls don't include a 2, so  $A$  and  $B$  can't both happen.

(b)

$A = \{\text{the first three rolls are all 6}\}$

$B = \{\text{the last three rolls are all 6}\}$

☐ disjoint    ☒ **independent**    ☐ neither

**Solution:** The first three rolls and the last three rolls are separate rolls of the die, so whether the first three are all 6 has no effect on whether the last three are.

(c)

$A = \{\text{you roll a 6 at least two times}\}$

$B = \{\text{you roll a 6 at least three times}\}$

☐ disjoint    ☐ independent    ☒ **neither**

**Solution:** Both can happen, since rolling a 6 at least three times means rolling a 6 at least two times. They aren't independent, since if  $B$  happens then  $A$  is certain to happen, so  $P(A | B) = 1$ , which is not equal to  $P(A)$ .

(d)

$A = \{\text{you roll a 6 at least two times}\}$

$B = \{\text{you roll a 1 at least two times}\}$

☐ disjoint    ☐ independent    ☒ **neither**

**Solution:** Both can happen, for example with the rolls 6, 6, 1, 1, 2, 3. But they aren't independent: if you roll a 6 at least twice, then there are at most four rolls left that could be 1, so rolling a 1 at least twice becomes less likely.

6. A box contains 5 yellow, 6 green, and 7 red jelly beans.

[4 points]

- (a) If 8 jelly beans are chosen randomly from the 18 jelly beans in the box, what is the probability that 3 are yellow and 3 are green?

**Solution:** Use sampling without replacement where order is irrelevant. There are  $\binom{18}{8}$  outcomes. The number of outcomes with 3 yellow, 3 green, and 2 red beans is

$$\binom{5}{3} \binom{6}{3} \binom{7}{2}.$$

So the probability is

$$\frac{\binom{5}{3} \binom{6}{3} \binom{7}{2}}{\binom{18}{8}}.$$

[5 points]

- (b) Suppose instead that only 3 jelly beans are chosen randomly from the 18 in the box. Let  $X$  be the number out of these 3 jelly beans that are red. Give the probability mass function (pmf) of  $X$ . **Note: this problem is not related to part (a).**

**Solution:** We use sampling without replacement where order is irrelevant, so there are  $\binom{18}{3}$  equally likely outcomes. The possible values of  $X$  are 0, 1, 2, and 3. For  $X = k$ , we choose  $k$  of the 7 red jelly beans and  $3 - k$  of the 11 jelly beans that aren't red. So,

$$P(X = 0) = \frac{\binom{7}{0}\binom{11}{3}}{\binom{18}{3}},$$

$$P(X = 1) = \frac{\binom{7}{1}\binom{11}{2}}{\binom{18}{3}},$$

$$P(X = 2) = \frac{\binom{7}{2}\binom{11}{1}}{\binom{18}{3}},$$

$$P(X = 3) = \frac{\binom{7}{3}\binom{11}{0}}{\binom{18}{3}}.$$

(These work out to  $\frac{165}{816}$ ,  $\frac{385}{816}$ ,  $\frac{231}{816}$ , and  $\frac{35}{816}$ , but you don't need to compute them.)

- [4 points] 7. You flip a coin three times. Given that the number of tails in the three flips is odd, what's the probability that all three flips are tails?

**Solution:** The sample space consists of the  $2^3 = 8$  equally likely sequences of three flips. Let  $A$  be the event that all three flips are tails, and  $B$  the event that the number of tails is odd. The number of tails is odd if it is 1 or 3, so

$$B = \{HHT, HTH, THH, TTT\},$$

and  $A \cap B = \{TTT\}$ . So,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/8}{4/8} = \frac{1}{4}.$$