1. Consider the following (false) statement about real numbers $x$ and $y$:

If $x < y$, then $\frac{1}{x} > \frac{1}{y}$.

Give a substitution for $x$ and $y$ that is a counterexample, and give a substitution for $x$ and $y$ that is not a counterexample.

**Solution:** $x = -1$ and $y = 1$ is a counterexample, since $-1 < 1$ but $\frac{1}{-1} \neq \frac{1}{1}$.

There are lots of different ways to have a noncounterexample. One example would be $x = 1$ and $y = -1$, so that the hypothesis is false.

2. Write the following statements using quantifiers as fully as possible.

(a) $t$ can be written as the sum of squares of four integers. (Here $t$ is a natural number.)

**Solution:** $(\exists a, b, c, d \in \mathbb{Z})(t = a^2 + b^2 + c^2 + d^2)$

(b) $n$ has two distinct divisors other than 1 and itself. (Here $n$ is a natural number.)

**Solution:** $(\exists a, b \in \mathbb{Z})(a|n \text{ and } b|n \text{ and } a \neq 1 \text{ and } a \neq n \text{ and } b \neq 1 \text{ and } b \neq n \text{ and } a \neq b)$

This question was a bit vague in the meaning of “divisor”: is a divisor of $n$ supposed to be a natural number or just an integer? So it would also be fine to say $(\exists a, b \in \mathbb{N})$.

3. Express the negations of each of the following statements using quantifier notation as fully as possible, but without using the negation symbol $\sim$ (but you can use symbols like $\notin$, $\not|$,$\neq$, etc.).

(a) $u$ is rational. (Here $u$ is a real number. The definition of a number being rational is that it can be written as an integer divided by another integer.)

**Solution:** $(\forall a, b \in \mathbb{Z})(u \neq \frac{a}{b})$

(b) $f$ is injective. (Here $f$ is a function from set $A$ to set $B$.)

**Solution:** $(\exists x, y \in A)(f(x) = f(y) \text{ and } x \neq y)$

(c) $f$ is surjective. (Here $f$ is a function from set $A$ to set $B$.)

**Solution:** $(\exists y \in B)(\forall x \in A)(f(x) \neq y)$

4. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$. In the following problems, you can describe the functions you come up with in any way you’d like (e.g., you can give an arrow diagram, or you can write out where each element of $A$ is mapped to by your function).
(a) Give an example of a function from $A$ to $B$ that is injective, or explain why no such example exists.

Solution: Define $f: A \rightarrow B$ by $f(1) = 1$, $f(2) = 2$, and $f(3) = 3$. This function is injective because you can't find two distinct elements of $A$ that are mapped to the same element of $B$.

(b) Give an example of a function from $A$ to $B$ that is surjective, or explain why no such example exists.

Solution: No such example exists because $B$ has 5 elements and $A$ only has 3. For any function $f: A \rightarrow B$, there will be at least two elements in $B$ that $f$ fails to map to.

(c) Give an example of a function from $A$ to $B$ that is neither injective nor surjective, or explain why no such example exists.

Solution: Define $f: A \rightarrow B$ that maps all elements of $A$ to 1, for example. It's not injective because $f(1) = f(2) = f(3)$. And it's not surjective because $f$ doesn't map anything to 2, 3, 4, or 5.

5. Let $X$ and $Y$ be sets and $f: X \rightarrow Y$. For a given set $B$ that is a subset of $Y$, we define

$$f^{-1}(B) = \{ x \in X : f(x) \in B \}.$$

(a) Suppose that $g: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $g(x) = x^2$. Let $A = \{-1\}$, $B = \{4\}$, and $C = \{0,4\}$. Find $g^{-1}(A)$, $g^{-1}(B)$, and $g^{-1}(C)$.

Solution: $g^{-1}(A) = \emptyset$, $g^{-1}(B) = \{-2,2\}$, and $g^{-1}(C) = \{-2,0,2\}$.

(b) Suppose that $f: X \rightarrow Y$ and $A,B \subseteq Y$. Prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Solution: I'll just do the proof that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. First, we show that $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$. Let $x \in f^{-1}(A \cup B)$. Then $f(x) \in A \cup B$. Hence $f(x) \in A$ or $f(x) \in B$. If $f(x) \in A$, then $x \in f^{-1}(A)$. If $f(x) \in B$, then $x \in f^{-1}(B)$. So, either $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$, which proves that $x \in f^{-1}(A) \cup f^{-1}(B)$. This completes the proof that $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$.

Now we prove that $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$. Let $x \in f^{-1}(A) \cup f^{-1}(B)$. Then $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. In the first case, $f(x) \in A$, and in the second case, $f(x) \in B$. Hence $f(x) \in A \cup B$, which implies that $x \in f^{-1}(A \cup B)$.

(c) Suppose that $f: X \rightarrow Y$ and $B \subseteq Y$. Does either of the following statements hold?

$$f(f^{-1}(B)) \subseteq B$$
and
\[ f(f^{-1}(B)) \supseteq B \]

Give a proof or counterexample for each statement.

Solution: It’s always true that \( f(f^{-1}(B)) \subseteq B \), and we’ll prove it now. Let \( y \in f(f^{-1}(B)) \). We must show that \( y \in B \). Since \( y \in f(f^{-1}(B)) \), there exists some \( x \in f^{-1}(B) \) such that \( f(x) = y \). It follows from \( x \in f^{-1}(B) \) that \( f(x) \in B \). But \( f(x) = y \), so we’ve shown that \( y \in B \).

The other statement is not always true. You should give a concrete counterexample, but I’m going to give you an abstract one. Consider any example of a nonsurjective function \( f : X \rightarrow Y \), and let \( B = Y \). Then \( f^{-1}(B) = X \) (why is this true?), and \( f(f^{-1}(B)) = f(X) \subseteq Y \) (why is this true?). Hence \( f(f^{-1}(B)) \) does not contain \( B \).

6. Let \( A \) and \( B \) be sets. Prove the following statement: There exists a function from \( A \) to \( B \) that is injective but not surjective if and only if there exists a function from \( B \) to \( A \) that is surjective but not injective. Note: This one would probably be too tricky to put on an actual final, but it’s a great question that will give you plenty of practice! Try drawing an arrow diagram.

Solution: When I started to write the solution, I noticed that this problem requires the assumption that \( A \) and \( B \) are nonempty! So let’s add that assumption. Also, let me make a note that this proof is really hard to read! But if you draw an arrow diagram, the idea is much simpler!

First, we show that if there exists a function from \( A \) to \( B \) that is injective but not surjective, then there exists a function from \( B \) to \( A \) that is surjective but not injective. Let \( f : A \rightarrow B \) be injective but not surjective. We must prove the existence of a function \( g : B \rightarrow A \) that is surjective but not injective.

Let \( U = f(A) \), which is a subset of \( B \). Observe that for any \( b \in U \), the set \( f^{-1}(b) \) contains exactly one element (see the previous question for the definition of \( f^{-1}(b) \)). This is because this set must contain at least one element since \( b \in f(A) \), and it contains no more than one element because \( f \) is injective.

Now, we define \( g : B \rightarrow A \) as follows. First, let \( a \) be some element of \( A \) (which exists since we’re assuming \( A \) is nonempty). For each \( b \notin U \), we define \( g(b) = a \). For each element \( b \in U \), we define \( g(b) \) to be equal to the single element in \( g^{-1}(b) \). This is a valid definition of a function (since we’ve mapped each element of \( B \) to a unique element of \( A \)), and we claim that \( g \) is surjective but not injective.

For the claim that it’s surjective, consider any element \( x \in A \). Let \( y = f(x) \). From the way we defined \( g \), we have \( g(y) = x \). Thus we’ve shown that for all elements \( x \in A \), there exists some \( y \in B \) such that \( g(y) = x \), proving that \( g \) is surjective.

For the claim that \( g \) is not injective, recall the element \( a \in A \) we used in the definition of \( g \). Let \( y = f(a) \). From the way we defined \( g \), we have \( g(y) = a \). Now, since \( f \) is not surjective, \( U \not\subseteq B \).
Let $y' \in B - U$. Then $y' \notin U$, and from our definition of $g$ we have $g(y') = a$. And $y' \neq y$ since $y \in U$ and $y' \notin U$. Since $g$ is not injective.

Now, we must show that if there exists a function from $B$ to $A$ that is surjective but not injective, then there exists a function from $A$ to $B$ to that is injective but not surjective. (I’ll finish this proof later).

7. Let $A$, $B$, and $C$ be sets, and let $f: A \to B$ and $g: B \to C$. For the following statements, give a proof if true and a counterexample if not.

(a) If $g \circ f$ is injective, then $f$ must be injective.
(b) If $g \circ f$ is injective, then $g$ must be injective.
(c) If $g \circ f$ is surjective, then $f$ must be surjective.
(d) If $g \circ f$ is surjective, then $g$ must be surjective.
(e) If $f$ is injective, then $g \circ f$ is injective.
(f) If $g$ is surjective, then $g \circ f$ is surjective.

**Solution:** This is not a full solution, but (a) and (d) are true and (b), (c), (e), and (f) are false.

**Proof of (a).** We’ll prove the contrapositive. Suppose that $f$ is not injective. Then there exists $u, v \in A$ such that $u \neq v$ and $f(u) = f(v)$. Since $f(u) = f(v)$, we have $g(f(u)) = g(f(v))$. So, we’ve shown that there exist $u, v \in A$ such that $u \neq v$ and $g(f(u)) = g(f(v))$, thus proving that $g \circ f$ is not injective.

**Counterexample for (b).** Let $A = [0, \infty)$, let $B = \mathbb{R}$, and let $C = [0, \infty)$. Define $f(x): A \to B$ by $f(x) = x$. Define $g: B \to C$ by $g(x) = |x|$. Then $g(f(x)) = x$, which is injective. But $g$ is not injective, since $g(-1) = g(1)$.

8. Write out the following statement using the $\sum$ symbol:

$$1(2) + 2(2)^2 + 3(2)^3 + 4(2^4) + \cdots + n(2)^n = (n-1)2^{n+1} + 2.$$  

Now prove the statement.

**Solution:** See the notes and video from the final class.

9. Show that $n^3 \leq 3^n$ for $n \geq 3$. 

**Solution:** When $n = 3$, this statement is true, since both of its sides are $3^3$.

Now, suppose that $t^3 \leq 3^t$ for some $t \geq 3$. We must show that $(t + 1)^3 \leq 3^{t + 1}$.

First, we claim that if $t \geq 3$, then $(t + 1)^3 \leq 3t^3$. To prove this, we will show that $t + 1 \leq 3^{1/3}t$, which is enough since then we can cube both sides to show that $(t + 1)^3 \leq 3t^3$. We can rearrange $t + 1 \leq 3^{1/3}t$ to get the equivalent statement

$$(3^{1/3} - 1)t \geq 1. \quad (1)$$

And now, we prove (1) by observing that if $t \geq 3$, then

$$(3^{1/3} - 1)t \geq (3^{1/3} - 1)3 \approx 1.32674 \geq 1.$$ 

So, we’ve now proven that if $t \geq 3$, then $(t + 1)^3 \leq 3t^3$.

Now, we multiply both sides of the inductive hypothesis by 3 to get

$$3t^3 \leq 3(3^t) = 3^{t + 1}.$$ 

Combined with the statement $(t + 1)^3 \leq 3t^3$, this shows that $(t + 1)^3 \leq 3^{t + 1}$. And this is what we were trying to prove.

By induction, $n^3 \leq 3^n$ for all $n \geq 3$.

10. Show that $5^n + 2(11)^n$ is divisible by 3. (*Hint:* $5^{n+1} + 2(11)^{n+1} = 5(5^n + 2(11)^n) + 12(11)^n$.)

**Solution:** The problem didn’t say for which values of $n$ we should prove this for, but we’ll do it all integers $n \geq 0$. So we start by checking that 3 divides $5^n + 2(11)^n$ when $n = 0$.

Now, we assume that $5^t + 2(11)^t$ is divisible by 3 for some nonnegative integer $t$, and we prove that $5^{t+1} + 2(11)^{t+1}$ is divisible by 3. We do this by rewriting $5^{t+1} + 2(11)^{t+1}$ as

$$5^{t+1} + 2(11)^{t+1} = 5(5^t) + 22(11)^t$$

$$= 5(5^t) + 10(11)^t + 12(11)^t$$

$$= 5(5^t + 2(11)^t) + 12(11)^t.$$

(*You might wonder how anyone would come up with this manipulation. Of course, the answer is that the exploration process that would get you there is not visible in the final proof! This is a common story in math.*) By the inductive hypothesis, $5^t + 2(11)^t$ is divisible by 3. Thus, $5(5^t + 2(11)^t)$ is divisible by 3. And since 12 is divisible by 3, the expression $12(11)^t$ is divisible by 3. Now, we’ve expressed $5^{t+1} + 2(11)^{t+1}$ as a sum of two multiples of 3, from which we can conclude that it is a multiple of 3 itself.

By induction, $5^n + 2(11)^n$ is divisible by 3 for all integers $n \geq 0$. 

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