Name: ____

Note: When transforming expressions into equivalent forms, write a chain of equalities. Don't just write one expression after the next without stating the relationship between the expressions.

When solving equations, write down one equation after another. Do not cross out or write on top of an equation.

1. Determine the radius of convergence for the series

$$\sum_{n=1}^{\infty} \frac{x^n}{2^n n^2} = \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{72} + \cdots .$$

Note: You do not need to determine the behavior of the series on the endpoints of the interval of convergence.

Solution: Apply the ratio test:

$$\rho = \left| \lim_{n\to\infty} \frac{\frac{x^{n+1}}{2^{n+1}(n+1)^2}}{\frac{x^n}{2^n n^2}} \right| = \lim_{n\to\infty} \left|\frac{x}{2}\right| \left(\frac{n}{n+1}\right)^2 = \left|\frac{x}{2}\right| \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^2 = \left|\frac{x}{2}\right|.$$

So, the series converges if $|x/2| < 1$ and diverges if $|x/2| > 1$. And the condition $|x/2| < 1$ is equivalent to $-2 < x < 2$. So, the radius of convergence for the series is 2.

There is another question on the back!

2. Compute the degree-3 Taylor polynomial for $f(x) = \sin x$ centered at $\pi/6$. It's of the form

$$A + B\left(x - \frac{\pi}{6}\right) + C\left(x - \frac{\pi}{6}\right)^2 + \left(x - \frac{\pi}{6}\right)^3,$$

and your job is to find A, B, C, and D, which should be (exact) numbers.

Note that $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

Solution: We have

$$\begin{aligned} f'(x) &= \cos x, \\ f''(x) &= -\sin x, \\ f^{(3)}(x) &= -\cos x. \end{aligned}$$

We plug in the center:

$$\begin{aligned} f(x) &= \frac{1}{2} \\ f'(x) &= \frac{\sqrt{3}}{2}, \\ f''(x) &= -\frac{1}{2}, \\ f^{(3)}(x) &= -\frac{\sqrt{3}}{2}. \end{aligned}$$

The Taylor polynomial is then

$$\frac{1}{2} + \frac{\sqrt{3}}{2}(x - \pi/6) - \frac{1}{4}(x - \pi/6)^2 - \frac{\sqrt{3}}{12}(x - \pi/6)^3.$$