

*I pledge that I have neither given nor received
unauthorized assistance during this examination.*

Signature:

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- When transforming expressions into equivalent forms, write a chain of equalities. Don't just write one expression after the next without stating the relationship between the expressions.
- When solving equations, write down one equation after another. Do not cross out or write on top of an equation.
- You can use a calculator, graphing or otherwise. You cannot use a computer or phone.
- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 7 problems on 7 pages.

Question	Points	Score
1	8	
2	6	
3	8	
4	8	
5	9	
6	6	
7	8	
Total:	53	

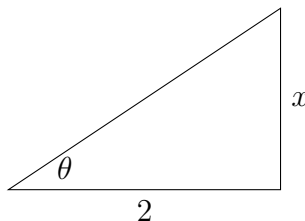
Good luck!

[8 points] 1. Compute $\int \frac{1}{(4+x^2)^{3/2}} dx$.

Solution: Make the substitution $x = 2 \tan \theta$. Then $dx = 2 \sec^2 \theta d\theta$, and the substitution gives

$$\begin{aligned} \int \frac{1}{(4+x^2)^{3/2}} dx &= \int \frac{2 \sec^2 \theta}{(4+4 \tan^2 \theta)^{3/2}} d\theta = \int \frac{2 \sec^2 \theta}{(4 \sec^2 \theta)^{3/2}} d\theta \\ &= \int \frac{2 \sec^2 \theta}{8 \sec^3 \theta} d\theta \\ &= \frac{1}{4} \int \cos \theta d\theta = \frac{1}{4} \sin \theta + C. \end{aligned}$$

Now, we have to work out $\sin \theta$ in terms of x . We know that $\tan \theta = x/2$. So we make a right triangle with an angle θ whose tangent is $x/2$, like this:



The other side has length $\sqrt{4+x^2}$ by the Pythagorean theorem. And this makes $\sin \theta = x/\sqrt{4+x^2}$. So,

$$\int \frac{1}{(4+x^2)^{3/2}} dx = \frac{x}{4\sqrt{4+x^2}} + C.$$

[6 points] 2. Compute $\int \frac{2x+3}{x^2+2x} dx$.

Solution: Factor $x^2 + 2x$ as $x(x+2)$ and set

$$\frac{2x+3}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}.$$

Now multiply both sides by $x(x+2)$ to get

$$2x+3 = A(x+2) + Bx.$$

Set $x = -2$ to get $-1 = -2B$, giving $B = 1/2$. And set $x = 0$ to get $3 = 2A$, giving $A = \frac{3}{2}$. So,

$$\int \frac{2x+3}{x^2+2x} dx = \int \left(\frac{3/2}{x} + \frac{1/2}{x+2} \right) dx = \frac{3}{2} \ln|x| + \frac{1}{2} \ln|x+2| + C.$$

- [8 points] 3. Compute the following series or state that they do not converge. Explain your answer. If you apply a test, you must give all details of the test to get full credit. (For example, for the comparison test, say what series you're comparing to.)

(a)
$$\sum_{n=0}^{\infty} \frac{3^n}{4^{n+2}}$$

Solution: This is a geometric series with first term $1/16$ and multiplier $3/4$. So it converges to $\frac{1/16}{1-3/4} = \frac{1}{4}$.

(b)
$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^2$$

Solution: The series diverges. One way to see this is that $\left(1 + \frac{1}{n}\right)^2 \geq 1$, and the series $\sum_{n=1}^{\infty} 1$ diverges. Another way is to observe that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^2 = 1$, which is not equal to zero, showing that the series diverges.

[8 points] 4. Compute the following improper integrals (or state that they diverge).

(a) $\int_0^{\infty} \frac{x}{(x^2 + 7)^3} dx$

Solution: We start by making the substitution $u = x^2 + 7$, which makes $du = 2x dx$. The integral transforms to

$$\frac{1}{2} \int_7^{\infty} \frac{du}{u^3}.$$

Now we compute

$$\frac{1}{2} \int_7^R \frac{du}{u^3} = \frac{1}{2} \left(-\frac{1}{2u^2} \right) \Big|_7^R = -\frac{1}{4R^2} + \frac{1}{196}.$$

Now

$$\int_0^{\infty} \frac{x}{(x^2 + 7)^3} dx = \frac{1}{2} \int_7^{\infty} \frac{du}{u^3} = \lim_{R \rightarrow \infty} \left(-\frac{1}{4R^2} + \frac{1}{196} \right) = \frac{1}{196}.$$

(b) $\int_0^{\infty} e^{-3x} dx$

Solution: First,

$$\int_0^R e^{-3x} dx = -\frac{1}{3} e^{-3x} \Big|_0^R = -\frac{1}{3} e^{-3R} + \frac{1}{3}.$$

Now,

$$\int_0^{\infty} e^{-3x} dx = \lim_{R \rightarrow \infty} \int_0^R e^{-3x} dx = \lim_{R \rightarrow \infty} \left(-\frac{1}{3} e^{-3R} + \frac{1}{3} \right) = \frac{1}{3},$$

since $e^{-3R} \rightarrow 0$ as $R \rightarrow \infty$.

[9 points] 5. Compute the following integrals:

(a) $\int x \ln x \, dx$

Solution: Do integration by parts with $u = \ln x$ and $dv = x \, dx$. Then $du = \frac{1}{x} \, dx$ and $v = x^2/2$, and

$$\begin{aligned} \int x \ln x \, dx &= \frac{x^2 \ln x}{2} - \frac{1}{2} \int x^2 \left(\frac{1}{x} \right) dx \\ &= \frac{x^2 \ln x}{2} - \frac{1}{2} \int x \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C. \end{aligned}$$

(b) $\int_0^2 \frac{1}{\sqrt{x+1}} \, dx$

Solution: Substitute $u = x + 1$ to get

$$\int_0^2 \frac{1}{\sqrt{x+1}} \, dx = \int_1^3 \frac{1}{\sqrt{u}} \, du = 2\sqrt{u} \Big|_1^3 = 2\sqrt{3} - 2.$$

(c) $\int x e^{2x} \, dx$

Solution: Do integration by parts with $u = x$ and $dv = e^{2x} \, dx$. Then $du = dx$ and $v = \frac{1}{2} e^{2x}$, and

$$\int x e^{2x} \, dx = \frac{x e^{2x}}{2} - \frac{1}{2} \int e^{2x} \, dx = \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + C$$

- [6 points] 6. Find the partial fractions decomposition of $\frac{x^2 - 9x + 22}{(x + 1)(x - 3)^2}$. (You do not have to integrate it.)

Solution: Set

$$\frac{x^2 - 9x + 22}{(x + 1)(x - 3)^2} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2}$$

and multiply both sides by $(x + 1)(x - 3)^2$ to get

$$x^2 - 9x + 22 = A(x - 3)^2 + B(x + 1)(x - 3) + C(x + 1)$$

Substitute $x = 3$ to get $4 = 4C$, yielding $C = 1$. Substitute $x = -1$ to get $32 = 16A$, yielding $A = 2$. Now substitute $x = 0$ (or anything else) to get

$$22 = 9A - 3B + C = 18 - 3B + 1,$$

yielding $B = -1$. So the decomposition is

$$\frac{x^2 - 9x + 22}{(x + 1)(x - 3)^2} = \frac{2}{x + 1} - \frac{1}{x - 3} + \frac{1}{(x - 3)^2}.$$

- [8 points] 7. State whether the following series converge or diverge. Explain your answer. If you apply a test, you must give all details of the test to get full credit. (For example, for the comparison test, say what series you're comparing to.)

(a)
$$\sum_{n=1}^{\infty} \frac{n+1}{n^2+2n+2}$$

Solution: There are different ways to approach this problem, but the easiest is probably to compare the series to $\sum_{n=1}^{\infty} \frac{1}{n}$ via the use the limit comparison test. We compute

$$\lim_{n \rightarrow \infty} \frac{\frac{n+1}{n^2+2n+2}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^2+n}{n^2+2n+2} = \lim_{n \rightarrow \infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}+\frac{2}{n^2}} = 1.$$

Since this limit is equal to 1, the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2+2n+2}$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{n}$ does. And we know that this second series does not converge.

(b)
$$\sum_{n=1}^{\infty} \frac{1}{n^2 10^n}$$

Solution: Since $\frac{1}{n^2 10^n} \leq \frac{1}{10^n}$ for all $n \geq 1$, and the series $\sum_{n=1}^{\infty} \frac{1}{10^n}$ converges (since it's geometric with multiplier $1/10$), this series converges too by the comparison test.