1. Let $f(x) = \frac{1}{\sqrt{1-x}}$.

   (a) Find $f'(x)$, $f''(x)$, and $f^{(3)}(x)$. Look for a pattern.

   Solution: Since $f(x) = (1-x)^{-1/2}$,

$$f'(x) = \frac{1}{2}(1-x)^{-3/2},$$
$$f''(x) = \frac{1}{2} \cdot \frac{3}{2}(1-x)^{-5/2}$$
$$f^{(3)}(x) = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}(1-x)^{-7/2}$$

   (b) Find a formula for $f^{(n)}(x)$, the nth derivative of $f(x)$.

   Solution: Each time we take another derivative, the exponent of $(1-x)$ goes down by 1, we multiply by 1/2, and we multiply by the next odd number. So,

$$f^{(n)}(x) = \frac{1(3)(5)\cdots(2n-1)}{2^n}(1-x)^{-(2n+1)/2}.$$

   (c) Write down the Taylor series for $f(x)$ centered at $x = 0$.

   Solution: Evaluating the derivatives at $x = 0$, we have

$$f^{(n)}(x) = \frac{1(3)(5)\cdots(2n-1)}{2^n} \text{ for any } n \geq 1.$$

   We also need to know the function evaluated at $x = 0$, which is $f(x) = 1$. So, the Taylor series is

$$\begin{aligned} T(x) &= 1 + \frac{1}{2}x + \frac{1(3)}{2^2 2!}x^2 + \frac{1(3)(5)}{2^3 3!}x^3 + \cdots + \frac{1(3)(5)\cdots(2n-1)}{2^n n!}x^n + \cdots \\ &= 1 + \sum_{n=1}^{\infty} \frac{1(3)(5)\cdots(2n-1)}{2^n n!}x^n. \end{aligned}$$

   It is not too important, but there is a nice way to simplify this expression. When you multiply all the odd numbers from 1 to $2n-1$, we can view this as multiplying *all* the numbers from 1 to $2n$ and then dividing by the even numbers from 2 to $2n$. All the even numbers from 2 to $2n$ multiplied together are

$$2(4)(6)\cdots 2n = 2^n(1)(2)\cdots(n) = 2^n n!.$$

   So, the odd numbers from 1 to $2n-1$ multiplied together are equal to

$$\frac{(2n)!}{2^n n!}.$$

The Taylor series is then

$$T(x) = 1 + \sum_{n=1}^{\infty} \frac{(2n)!}{2^n n!} \frac{1}{2^n n!} x^n = 1 + \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} x^n = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} x^n.$$

Note: I'm not asking you to show this, but let me note that the radius of convergence of this Taylor series is 1 (you could show this with the ratio test) and that on $(-1, 1)$, the Taylor series does indeed converge to $f(x)$ (you could show this using the Taylor series error bound).

(d) By substitution into the previous Taylor series, find the Taylor series for $\frac{1}{\sqrt{1-x^2}}$.

Solution: We just replace x with x^2 in the Taylor series:

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{n=1}^{\infty} \frac{1(3)(5)\cdots(2n-1)}{2^n n!} x^{2n}.$$

(e) Find the Taylor series for $\sin^{-1}(x)$. (Note that the derivative of $\sin^{-1}(x)$ is $\frac{1}{\sqrt{1-x^2}}$).

Solution: We can integrate term by term to get the Taylor series for $\sin^{-1}(x) + C$:

$$\sin^{-1} x + C = x + \sum_{n=1}^{\infty} \frac{1(3)(5)\cdots(2n-1)}{2^n n!(2n+1)} x^{2n+1}.$$

When we plug in $x = 0$, we get C on the left since $\sin^{-1} x = 0$, and we get 0 on the right. So $C = 0$ and the right-hand side is the Taylor series for $\sin^{-1} x$.

(f) Since $\sin^{-1}(1/2) = \pi/6$, we can plug $x = \frac{1}{2}$ into the Taylor series from the previous problem to get a sum converging to $\pi/6$. Write down the sum. *Note: this is one way of computing the value of π to as many decimal places as desired!*

Solution: When we plug in $x = 1/2$, the sum becomes

$$\frac{\pi}{6} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1(3)(5)\cdots(2n-1)}{2^{3n+1} n!(2n+1)}.$$