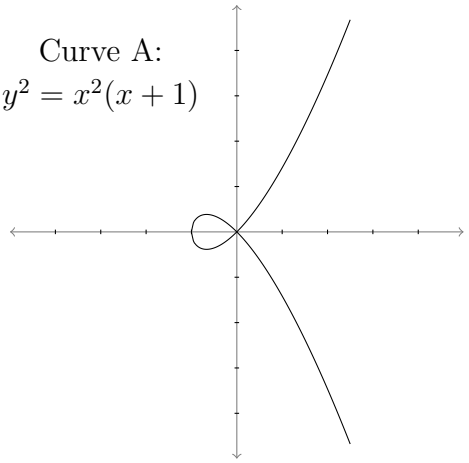
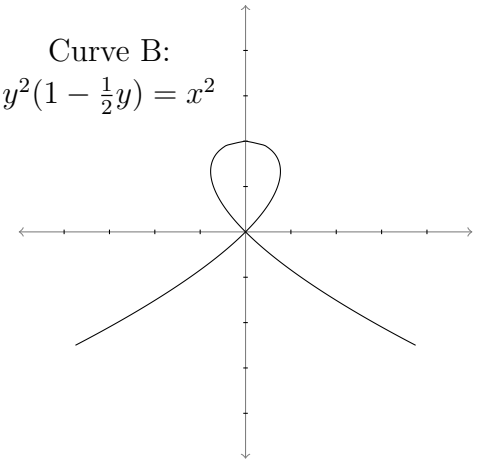
1. Here are two curves defined by equations:

Curve A:
$y^2 = x^2(x+1)$



Curve B:
$y^2(1 - \frac{1}{2}y) = x^2$



(a) A cat is at the following location at time t:

$$(x(t), y(t)) = (t^2 - 1, t - t^3).$$

This cat is following one of the two curves. Which one? Explain how you know.

Solution: Check that the expressions for $x(t)$ and $y(t)$ satisfy the equation corresponding to Curve A, not to Curve B. That is, $y(t)^2 = x(t)^2(x(t)+1)$.

(b) Which direction does the cat go on the curve? How do you know?

Solution: As long as t is bigger than 1, the expression $y(t) = t - t^3$ is negative, so the cat eventually stays in the lower half of the plane. The cat enters the picture in the upper right, runs around a loop near the origin, and runs away to the lower right.

(c) Find all points on the curve that have a horizontal tangent line. *Hint:* What will be true about $x'(t)$ and $y'(t)$ at a point with a horizontal tangent line?

Solution: The tangent line will be horizontal when $y'(t) = 0$. So, we solve

$$y'(t) = 1 - 3t^2 = 0,$$

which occurs when $t = \pm 1/\sqrt{3}$, when the cat is at points $(-2/3, 2/3\sqrt{3})$ and $(-2/3, -2/3\sqrt{3})$

(d) At which time(s) t does the cat run through the point $(0, 0)$?

Solution: If $x(t) = 0$, then $t = \pm 1$, and if $y = 0$ then $t = \pm 1$ or $t = 0$. So the only times when x and y are both zero are $t = \pm 1$.

(e) What are the tangent line(s) to the parametrized curve $(x(t), y(t))$ at $(0, 0)$?

Solution: The slope of the tangent line to the curve at $(x(t), y(t))$ is $y'(t)/x'(t)$. Now

$$x'(t) = 2t, \qquad y'(t) = 1 - 3t^2.$$

At $t = 1$ we have $y'(t) = -2$ and $x'(t) = 2$; at $t = -1$ we have $y'(t) = -2$ and $x'(t) = -2$. So the slopes are -1 and 1, respectively, and the equations of the tangent lines are

$$y = -x \qquad \text{and} \qquad y = x.$$

(f) What is the slowest speed of the cat at any time?

Solution: The speed of the cat at time t is

$$\sqrt{4t^2 + (1 - 3t^2)^2} = \sqrt{9t^4 - 2t^2 + 1}.$$

The goal is to minimize this expression. It's equivalent to minimizing

$$f(t) = 9t^4 - 2t^2 + 1,$$

which is easier to do since there's no square root. To find the minimum of this function, we find $f'(t)$ and set it equal to 0 and solve for t:

$$f'(t) = 36t^3 - 4t = 4t(9t^2 - 1) = 36t(t - 1/3)(t + 1/3).$$

So $f(t)$ has critical points at $t = -1/3$, $t = 0$, and $t = 1/3$. We plug in some points to find that $f'(t)$ is negative for $t < -1/3$, positive for $-1/3 < t < 0$, negative for $0 < t < 1/3$, and positive for $t > 1/3$. So local minima occur at $t = \pm 1/3$. The speed at those times is $\sqrt{8}/3$.

(g) Set up an integral that computes how far the cat goes when traversing the loop of the curve.

Solution: The question is to find the arc length of the curve from $t = -1$ to $t = 1$:

$$\int_{-1}^{1} \sqrt{(2t)^2 + (1 - 3t^2)^2}\, dt = \int_{-1}^{1} \sqrt{9t^4 - 2t^2 + 1}\, dt.$$