Basic differentiation

Covered in sections 3.3–3.7, 3.9.

1. Find derivatives of the following functions:

   (a) $f(\theta) = \sin(\ln\theta)$

   > **Solution:** $f'(\theta) = \cos(\ln\theta)\frac{1}{\theta}$

   (b) $h(t) = t^{(t^t)}$

   > **Solution:** There are several different ways to do this one. We'll use the notation $\exp(x) = e^x$. We can rewrite $h(t)$ as
   >
   > $$h(t) = \exp\Big((\ln t)t^t\Big) = \exp\Big((\ln t)\exp\big((\ln t)t\big)\Big).$$
   >
   > By the chain rule,
   >
   > $$h'(t) = \exp\Big((\ln t)\exp\big((\ln t)t\big)\Big)\frac{d}{dt}\Big((\ln t)\exp\big((\ln t)t\big)\Big).$$
   >
   > Now let's work on the last factor above. Using the product rule in the first line and the chain rule in the second,
   >
   > $$\begin{aligned}\frac{d}{dt}\Big((\ln t)\exp\big((\ln t)t\big)\Big) &= \frac{1}{t}\exp\big((\ln t)t\big) + (\ln t)\frac{d}{dt}\Big(\exp\big((\ln t)t\big)\Big)\\ &= \frac{1}{t}\exp\big((\ln t)t\big) + (\ln t)\exp\big((\ln t)t\big)\frac{d}{dt}\Big((\ln t)t\Big)\\ &= \frac{1}{t}\exp\big((\ln t)t\big) + (\ln t)\exp\big((\ln t)t\big)\left(\frac{1}{t}t + \ln t\right)\\ &= \frac{1}{t}\exp\big((\ln t)t\big) + (\ln t)\exp\big((\ln t)t\big)\big(1 + \ln t\big).\end{aligned}$$
   >
   > So, all together,
   >
   > $$\begin{aligned}h'(t) &= \exp\Big((\ln t)\exp\big((\ln t)t\big)\Big)\left(\frac{1}{t}\exp\big((\ln t)t\big) + (\ln t)\exp\big((\ln t)t\big)\big(1 + \ln t\big)\right)\\ &= t^{(t^t)}\left(\frac{1}{t}t^t + (\ln t)(1+\ln t)t^t\right)\\ &= t^{(t^t)}t^t\left(\frac{1}{t} + (\ln t)(1+\ln t)\right).\end{aligned}$$
   >
   > Another potential approach would have been to use logarithmic differentiation.

   (c) $f(x) = x^2e^{1/x}$

Solution:

$$f'(x) = 2xe^{1/x} + x^2e^{1/x}(-x^{-2}) = 2xe^{1/x} - e^{1/x}$$

2. Find equations for the tangent line to the graph of f at $x = a$:

   (a) $f(x) = x^2 - x, \quad a = 1$

   Solution: In these problems, the first step is to find $f'(a)$. The next step is to use the point-slope equation for a line to get the equation of a line with slope $f'(a)$ that contains the point $(a, f(a))$.
   Doing this, we find that $f'(x) = 2x - 1$ and $f'(1) = 1$. The tangent line contains the point $(a, f(a))$, or $(1, 0)$. So, the tangent line is

   $$y - 0 = 1(x - 1),$$

   or just $y = x - 1$.

   (b) $f(x) = 5 - 3x, \quad a = 2$

   Solution: Doing this one first without thinking, we find $f'(2) = -3$ and then we find the point $(a, f(a)) = (2, -1)$, giving us a line $y + 1 = -3(x - 2)$.
   With a bit more thought, the graph of $f(x)$ is a line, and so the tangent line to it is just the identical line, $y = 5 - 3x$. Note that this is equivalent to the equation above.

Limits (including L'Hôpital's rule)

Covered in sections 2.3–2.7 and 4.5.

3. Evaluate the limit or state that it doesn't exist.

   (a)

   $$\lim_{x\to\infty} \frac{x^2 - 3x^4}{x - 1}$$

   Solution:

   $$\lim_{x\to\infty} \frac{x^2 - 3x^4}{x - 1} = \lim_{x\to\infty} \frac{-3x^3 + x}{1 - \frac{1}{x}}$$

   The top of this diverges to $-\infty$ and the bottom converges to 1. So, the limit diverges to $-\infty$. (Saying that the limit doesn't exist is also valid.)

(b)

$$\lim_{x\to 1} x^{1/(x-1)}$$

Solution: Let $f(x) = x^{1/(x-1)}$. Then $\ln f(x) = \frac{\ln x}{x-1}$. By L'Hôpital's rule,

$$\begin{aligned}\lim_{x\to 1} \ln f(x) = \lim_{x\to 1} \frac{\ln x}{x-1} &= \lim_{x\to 1} \frac{\frac{1}{x}}{1} \\ &= 1.\end{aligned}$$

It follows from $\lim_{x\to 1} \ln f(x) = 1$ that $\ln\bigl(\lim_{x\to 1} f(x)\bigr) = 1$. Exponentiating each side of thise equation, we get

$$\lim_{x\to 1} f(x) = e^1 = e.$$

(c)

$$\lim_{x\to 0} \frac{x^3}{\sin x - x}$$

Solution: By L'Hôpital's rule applied three times,

$$\lim_{x\to 0} \frac{x^3}{\sin x - x} = \lim_{x\to 0} \frac{3x^2}{\cos x - 1} = \lim_{x\to 0} \frac{6x}{-\sin x} = \lim_{x\to 0} \frac{6}{-\cos x} = -6.$$

(d)

$$\lim_{x\to 3} \frac{2x^2 - 5x - 3}{x - 4}$$

Solution: By direct evaluation,

$$\lim_{x\to 3} \frac{2x^2 - 5x - 3}{x - 4} = \frac{2(3^2) - 5(3) - 3}{3 - 4} = \frac{18 - 15 - 3}{-1} = 0.$$

Implicit Differentiation

Covered in section 3.8.

4. Find an equation for the line tangent to the curve $x^2 + \sin y = xy^2 + 1$ at the point $(1, 0)$.

Solution: Use implicit differentiation, taking the derivative of both sides of the equation:

$$\frac{d}{dx}\Big(x^2 + \sin y\Big) = \frac{d}{dx}\Big(xy^2 + 1\Big)$$

Expanding both sides, we get

$$2x + (\cos y)\frac{dy}{dx} = y^2 + 2xy\frac{dy}{dx}.$$

Now, we plug in $x = 1$ and $y = 0$:

$$2 + \frac{dy}{dx} = 0.$$

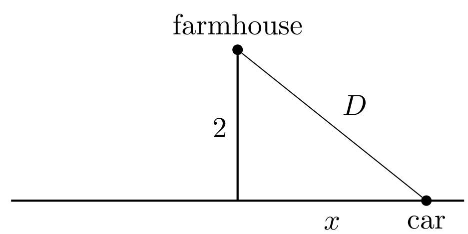
So, at $(1, 0)$, we have $\frac{dy}{dx} = -2$.

Related Rates

Covered in section 3.10.

5. A road perpendicular to a highway leads to a farmhouse located 2 km off the highway. An automobile travels on the highway at 80 km/h. How fast is the distance between the automobile and farmhouse increasing when the automobile is 6 km past the intersection of the highway and road?

Solution: Let's start by making a picture and assigning variable names:



We are given that $\frac{dx}{dt} = 80$. We want to find the value of $\frac{dD}{dt}$ when $x = 6$. We use the Pythagorean theorem to relate x and D:

$$D^2 = 4 + x^2.$$

We differentiate both sides of the equation to obtain

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt}.$$

When $x = 6$, we have $D = \sqrt{4+36} = \sqrt{40}$. Plugging all known values into the above equation,

$$2\sqrt{40}\frac{dD}{dt} = 2(6)(80).$$

Solving for $\frac{dD}{dt}$, we get

$$\frac{dD}{dt} = \frac{2(6)(80)}{2\sqrt{40}} \approx 75.89.$$

The final conclusion is that the distance between the car and the farmhouse is increasing at 75.89 km/h.

Linear Approximation

Covered in section 4.1.

6. Let $P = (2, 1)$, a point on the curve $y^3 + 3xy = 7$. Give the approximate y-coordinate of the point on the curve near P with x-coordinate 2.1.

Solution: We use implicit differentiation to find $\frac{dy}{dx}$ at P:

$$3y^2\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} = 0$$
$$\implies \frac{dy}{dx} = -\frac{y}{x+y^2} = -\frac{1}{3}.$$

Now, we use the linear approximation formula to estimate the y-value at 2.1, which is shifted over from P by $\Delta x = 0.1$:

$$y \approx x + \frac{dy}{dx}\Delta x = 1 - \frac{1}{3}(0.1) \approx 0.967.$$

Maxima, minima, and optimization

Covered in sections 4.2, 4.7.

7. Find the maximum value of $f(x) = 2\sqrt{x} - x$ on $[0, 4]$.

Solution: Because we're finding the maximum of a continuous function on a closed interval, we just need to check the function at critical points and at the endpoints. First, we find the critical points:

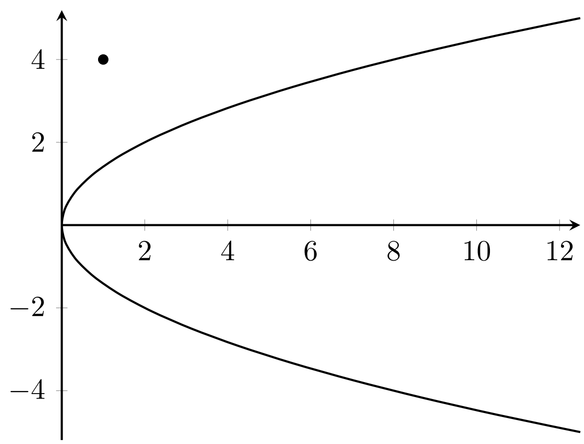
$$f'(x) = \frac{1}{\sqrt{x}} - 1 = 0,$$

yielding a single critical point at $x = 1$. We evaluate

$$f(0) = 0,$$
$$f(1) = 1$$
$$f(4) = 0.$$

So, the maximum value of $f(x)$ on the interval is 1, occurring at $x = 1$.

8. Find the point on the curve $y^2 = 2x$ closest to $(1, 4)$.



Solution: Let (x, y) be a point on the curve. Let D be the distance from (x, y) to $(1, 4)$, given by

$$D = \sqrt{(x-1)^2 + (y-4)^2}.$$

Our goal is to minimize D subject to the constraint $y^2 = 2x$ that arises since (x, y) is a point on the given curve.

Rewriting the constraint, we have $x = y^2/2$. Plugging this into the formula for D gives

$$D = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2}.$$

Ordinarily, our goal would be to maximize D on $(-\infty, \infty)$, and this approach will work. However, it's easier to consider the function D^2, given by

$$f(y) = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2$$

and then to maximize this instead. Since the maximum of D^2 occurs at the same place as the maximum of D, this is just as good.

First, we find all critical points of $f(y)$ by finding $f'(y)$, setting it equal to 0, and then solving for y. This will give $y = 2$. We can then check that $f'(y)$ is negative for $y < 2$ and is positive for $y > 2$. This means that an absolute minimum occurs at $y = 2$. So, the minimum point on the curve is when $y = 2$, in which case $x = 2^2/2 = 2$. Thus, the closest point to $(1, 4)$ on the curve is $(2, 2)$.

The shape of a graph

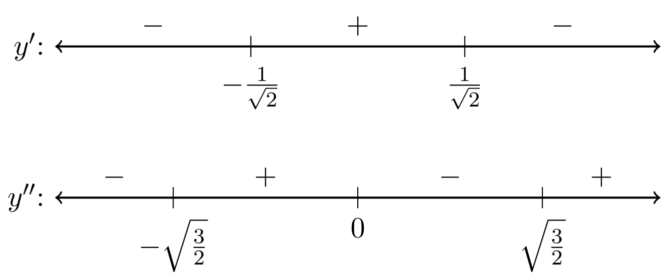
Covered in sections 4.3, 4.4, and 4.6.

9. Sketch the graph of $y = xe^{-x^2}$. Be accurate with regard to whether the graph is increasing or decreasing, its concavity, and its asymptotic behavior.

Solution: We need to determine when y' and y'' are positive and negative.

$$y' = e^{-x^2} - 2x^2e^{-x^2} = (1 - 2x^2)e^{-x^2},$$
$$y'' = -2xe^{-x^2} - 4xe^{-x^2} + 4x^3e^{-x^2} = 2(2x^2 - 3)xe^{-x^2}.$$

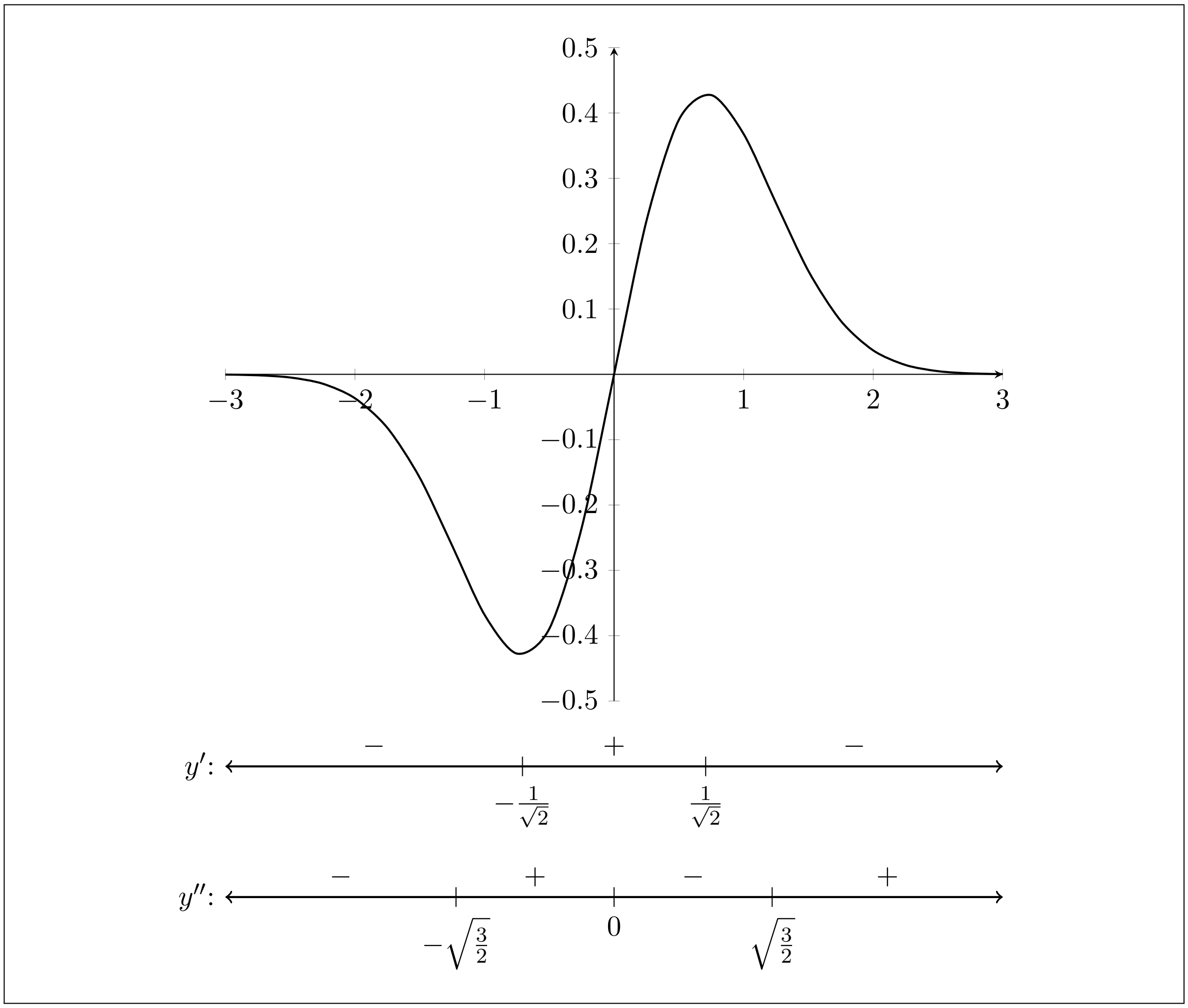
This gives us critical points $x = \pm 1/\sqrt{2}$, and it gives us roots of y'' of 0 and $\pm\sqrt{3/2}$. Since e^{-x^2} is always positive, we get the following sign charts for y' and y'':



The function has no vertical asymptototes. It has horizontal asymptotes in both directions at $y = 0$, since

$$\lim_{x\to\pm\infty} xe^{-x^2} = 0,$$

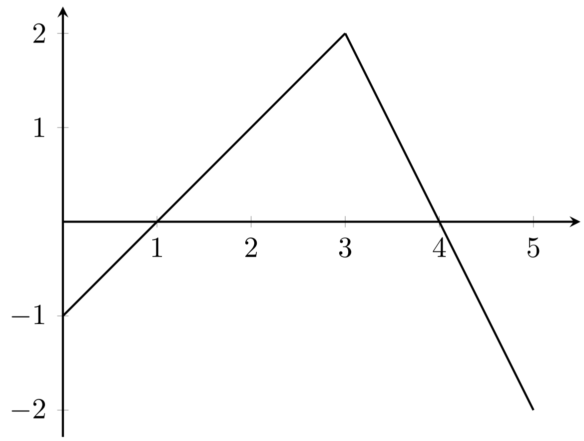
which you can confirm using L'Hôpital's rule. Here's a sketch taking all of this into consideration, with the sign diagrams underneath for easy reference:



Area and definite integrals

Covered in sections 5.1, 5.2.

10. The following is a graph of $y = g(x)$.



Evaluate $\int_0^5 g(t)\,dt$.

Solution: There are three areas to be computed: a triangle below the y-axis from $x = 0$ to $x = 1$, a triangle above the y-axis from $x = 1$ to $x = 4$, and a triangle below the y-axis from $x = 4$ to $x = 5$. The first has signed area $-\frac{1}{2}$, the next area 3, and the last area -1. So,

$$\int_0^5 g(t)\,dt = -\frac{1}{2} + 3 - 1 = 1.5.$$

11. Compute R_5, the right endpoint approximation with 5 rectangles, for the area under the curve $f(x) = x^2 + x$ from -1 to 1.

Solution: The width of each rectangle is $2/5 = .4$. The right endpoints of the rectangles are $-.6$, $-.2$, .2, .6, and 1. We evaluate

$$\begin{aligned} f(-.6) &= -.24, \\ f(-.2) &= -.16, \\ f(.2) &= .24, \\ f(.6) &= .96, \\ f(1) &= 2. \end{aligned}$$

So, the total signed area of the rectangles is

$$.4(-.24 - .16 + .24 + .96 + 2) = 1.12.$$

Antiderivatives, the fundamental theorem of calculus, and integration techniques

Covered in sections 5.3–5.5, 5.7–5.8.

12. Compute the following integrals.

(a)

$$\int (\sqrt{t}+1)(t+1)\,dt$$

Solution:

$$\frac{2}{5}t^{5/2}+\frac{1}{2}t^{2}+\frac{2}{3}t^{3/2}+t+C$$

(b)

$$\int_{-2}^{0}(3x-9e^{3x})dx$$

Solution: $3e^{-6}-9$

(c) Find

$$\frac{d}{dx}\int_{1}^{x^4}\sqrt{t}\,dt$$

Solution: $x^2\cdot 4x^3=4x^5$

(d)

$$\int_{0}^{1}\frac{x}{(x^2+1)^3}dx$$

Solution: $\frac{3}{16}$

(e)

$$\int \frac{1}{\sqrt{9-4x^2}}\,dx$$

Solution: $\frac{1}{2}\sin^{-1}\left(\frac{2}{3}x\right)+C$