Math 231, Midterm 2 Name: ____________________

April 17, 2024

I pledge that I have neither given nor received unauthorized assistance during this examination.

Signature:

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- You may use any sort of calculator, but not a phone or a computer.
- It is okay to leave a numerical answer like $\frac{39}{2} - (18 - e^2)$ unsimplified.
- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 7 problems on 10 pages.

Question	Points	Score
1	16	
2	12	
3	10	
4	10	
5	12	
6	8	
7	10	
Total:	78	

Good luck!

[16 points] 1. Find the derivatives of the following functions. Do **not** simplify your solutions.

(a) $f(t) = \dfrac{1+e^t}{1-e^t}$

Solution:

$$f'(t) = \frac{(1-e^t)e^t + (1+e^t)(-e^t)}{(1-e^t)^2}$$

(b) $f(x) = \ln\big(4x^2+1\big)$

Solution:

$$f'(x) = \frac{1}{4x^2+1}(8x)$$

(c) $f(\theta) = \cos(\cos(\cos\theta))$

Solution:

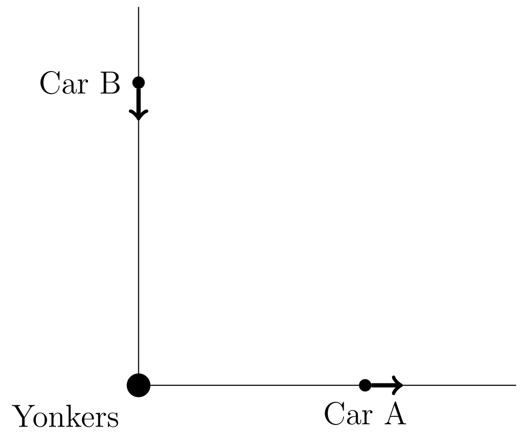
$$f'(\theta) = -\sin(\cos(\cos\theta))(-\sin(\cos\theta))(-\sin\theta)$$

(d) $f(x) = \sin(2x)\cos x$

Solution:

$$f'(x) = \cos(2x)(2)\cos x + \sin(2x)(-\sin x)$$

2. Car A drives east away from Yonkers at 30 mph. Car B drives south towards Yonkers at 10 mph. (See the diagram below.)



[2 points] (a) Let u be the distance between Car A and Car B. What is u at the moment when Car A is 3 miles east of Yonkers and Car B is 4 miles north of Yonkers?

Solution: Let x be the distance from car A to Yonkers and let y be the distance from car B to Yonkers. Then u is the length of the hypotenuse of a right triangle with sides x and y. When $x = 3$ and $y = 4$, we have $3^2 + 4^2 = u^2$ and hence $u = 5$.

[10 points] (b) What is the rate of change of u at the moment when Car A is 3 miles east of Yonkers and Car B is 4 miles north of Yonkers?

Solution:

Solution:
Assign variables: Let x be the distance from car A to Yonkers and let y be the distance from car B to Yonkers. Let u be the distance between the two cars, as given.

Known. We're given that $\frac{dx}{dt} = 30$ mph and $\frac{dy}{dt} = -10$ mph.

Want. We want to find $\frac{du}{dt}$ at the moment when $x = 3$ and $y = 4$.

Relation. By the Pythagorean theorem, $u^2 = x^2 + y^2$.

Take derivatives, substitute, solve. Taking the derivative of both sides of the relation gives

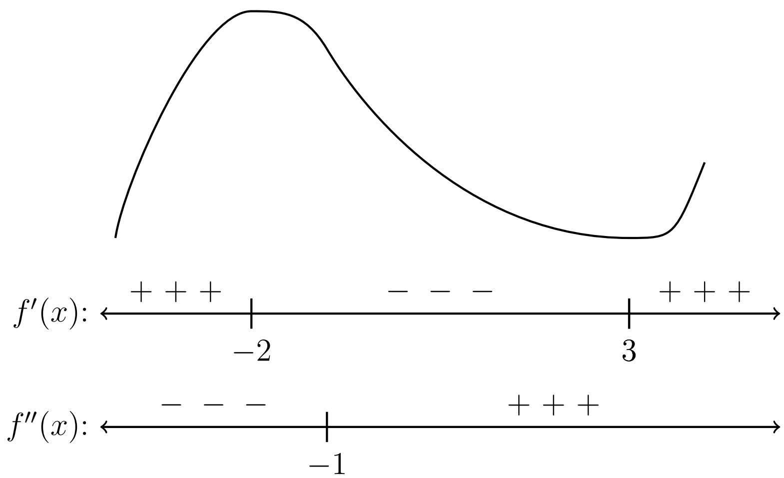
$$2u\frac{du}{dt} = 2x\frac{dx}{dt} + 2y\frac{2y}{dt}.$$

When $x = 3$ and $y = 4$, the value of u is 5 by part (a). Plugging in these values along with $\frac{dx}{dt} = 30$ and $\frac{dy}{dt} = -10$ gives

$$2(5)\frac{du}{dt} = 2(3)(30) + 2(4)(-10) = 180 - 80 = 100.$$

So, $\frac{du}{dt} = 10$. That is, the distance between the two cars is increasing at 10 mph.

3. Suppose that the signs of $f'(x)$ and $f''(x)$ are given below:



[2 points] (a) Give all x-coordinates where $f(x)$ has a local minimum, or state that there are none.

Solution: $x = 3$, since $f'(x)$ changes from negative to positive there.

[2 points] (b) Give all x-coordinates where $f(x)$ has a local maximum, or state that there are none.

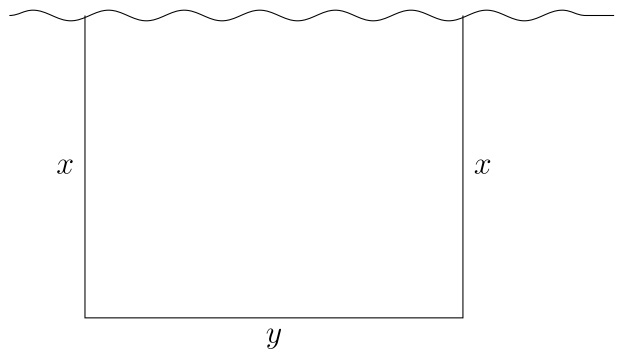
Solution: $x = -2$, since $f'(x)$ changes from positive to negative there.

[2 points] (c) Give all x-coordinates where $f(x)$ has an inflection point, or state that there are none.

Solution: $x = -1$, since $f''(x)$ changes signs there.

[4 points] (d) Sketch the graph in the space above the sign diagrams.

[10 points] 4. A rectangular area along the ocean is to be fenced off for a private event. The fence only needs to cover three sides of the area; the ocean is the fourth side. Suppose the fence parallel to the water has length y and the two sides of fence perpendicular to the water have length x as in the diagram.



Suppose that you have 80 meters of fencing in total that you will use to block off the area. What are the dimensions x and y of the rectangle with the largest area satisfying this contraint?

Solution: The constraint on the rectangle is that the three sides add up to 80 meters, i.e., $2x + y = 80$. The goal is to maximize the area xy. So the objective function is

$$A(x) = xy = x(80 - 2x) = 80x - 2x^2.$$

We are trying to maximize $A(x)$ over the interval $[0, 40]$ (the 40 is chosen because the two sides of length x use up all 80 meters of fencing when $x = 40$). We look for critical points of $A(x)$:

$$A'(x) = 80 - 4x = 0,$$

yielding a solution $x = 20$. We have $A(0) = 0$, $A(20) = 800$, and $A(40) = 0$. So the maximum occurs at $x = 20$. When $x = 20$, we have $y = 40$. So the dimensions of the rectangle maximizing the area are 20×40.

5. Let $f(x) = \dfrac{1}{4 - x^2}$.

[3 points] (a) Does this function have horizontal asymptotes at $\pm\infty$? If so, say what they are.

Solution: Since

$$\lim_{x \to \pm\infty} \frac{1}{4 - x^2} = 0,$$

this function has horizontal asymptotes $y = 0$ at $\pm\infty$.

[3 points] (b) Does this function have any vertical asymptotes? If so, say what they are.

Solution: This function has vertical asymptotes at x-values that make its denominator equal to 0. To find them, we solve $4 - x^2 = 0$ and get $x = \pm 2$. So the function has vertical asymptotes at $x = \pm 2$.

[3 points] (c) On what intervals is this function increasing?

Solution: Writing $f(x) = (4 - x^2)^{-1}$, we find

$$f'(x) = -\frac{1}{(4-x^2)^2}(-2x) = \frac{2x}{(4-x^2)^2}.$$

We have a critical point at $x = 0$, and we have the vertical asymptotes at $x = \pm 2$. So we need to figure out the direction of the function on the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$. The derivative is positive when $x > 0$ and negative when $x < 0$, since its numerator is positive when $x > 0$ and negative when $x < 0$ and its denominator is always positive. So the function is increasing on $(0, 2)$ and $(2, \infty)$, where the derivative is positive.

[3 points] (d) On what intervals is this function decreasing?

Solution: On $(-\infty, -2)$ and $(-2, 0)$, where the derivative is negative.

[8 points] 6. Consider the curve formed by the solutions to $x^3 + y^3 = 3xy$. What is the slope of its tangent line at the point $(\frac{2}{3}, \frac{4}{3})$?

Solution: Use implicit differentiation to obtain

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}.$$

Now solve for $\frac{dy}{dx}$, getting

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

Now plug in the point:

$$\frac{dy}{dx} = \frac{\frac{4}{3} - \frac{4}{9}}{\frac{16}{9} - \frac{2}{3}} = \frac{\frac{8}{9}}{\frac{10}{9}} = \frac{4}{5}.$$

7. Let $f(x) = 1 + (\sin x)(\cos x)$.

[2 points] (a) Find $f'(x)$.

Solution:

$$f'(x) = \cos^2 x - \sin^2 x.$$

[8 points] (b) Estimate the value of $f(0.01)$ using the linear approximation of $f(x)$ based at $x = 0$.

Solution: We have $f(0) = 1 + (0)(1) = 0$ and $f'(0) = 1 - 0 = 1$. According to the linear approximation formula,

$$f(0.01) \approx f(0) + (0.01)f'(0) = 1.01.$$