Math 231, Midterm 1, Version A Name: ____________________

March 4, 2024

I pledge that I have neither given nor received unauthorized assistance during this examination.

Signature:

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- You may use any sort of calculator, but not a phone or a computer.
- It is okay to leave a numerical answer like $\frac{39}{2} - (18 - e^2)$ unsimplified.
- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 6 problems on 9 pages.

Question	Points	Score
1	16	
2	13	
3	10	
4	16	
5	9	
6	10	
Total:	74	

Good luck!

[16 points] 1. (a)

$$\lim_{x\to\infty} \frac{3x^2+x-1}{2x^2+4} =$$

Solution:

$$\lim_{x\to\infty} \frac{3x^2+x-1}{2x^2+4} = \lim_{x\to\infty} \frac{3x^2+x-1}{2x^2+4}\left(\frac{1/x^2}{1/x^2}\right)$$
$$= \lim_{x\to\infty} \frac{3+\frac{1}{x}-\frac{1}{x^2}}{2+\frac{4}{x^2}} = \frac{3}{2}.$$

(b)

$$\lim_{x\to 1} \frac{x^2+3x-4}{x-1} =$$

Solution: Subsitution yields $\frac{0}{0}$, so we need to manipulate the expression to find the limit.

$$\lim_{x\to 1} \frac{x^2+3x-4}{x-1} = \lim_{x\to 1} \frac{(x+4)(x-1)}{x-1} = \lim_{x\to 1}(x+4) = 5.$$

(c)

$$\lim_{x\to 3} \frac{x^2+4}{x^2-9} =$$

Solution: The top of this fraction converges to 13 and the bottom to 0 as x approaches 3. Thus the limit diverges. (You don't need to work this out, but the limit from the left is $-\infty$ and the limit from the right is $+\infty$.

(d)

$$\lim_{x\to\infty} \sin\left(\frac{\pi(x^2+1)}{4x^2+1}\right) =$$

Note: you will get a bonus point on this problem if you give the exact answer rather than a numerical approximation.

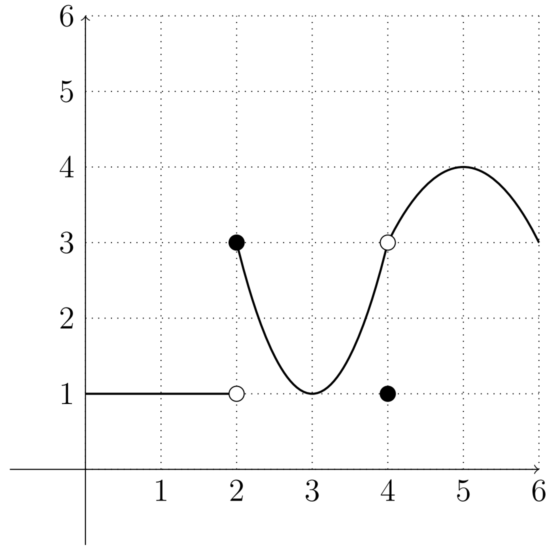
Solution: First, we find

$$\begin{aligned}\lim_{x\to\infty} \frac{\pi(x^2+1)}{4x^2+1} &= \pi \lim_{x\to\infty} \frac{x^2+1}{4x^2+1}\left(\frac{1/x^2}{1/x^2}\right) \\ &= \pi \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{4+\frac{1}{x^2}} = \frac{\pi}{4}.\end{aligned}$$

So,

$$\lim_{x\to\infty} \sin\left(\frac{\pi(x^2+1)}{4x^2+1}\right) = \sin\left(\lim_{x\to\infty} \frac{\pi(x^2+1)}{4x^2+1}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

[13 points] 2. Here is a graph of a function $f(x)$. Find the following quantities. If a limit does not exist, say so. (If a limit diverges to infinity, it is acceptable either to say that it doesn't exist, or to say that it diverges to infinity, or to write down that its limit is ∞ or $-\infty$.) You do not need to show any work on this problem.



(a) $\lim_{x\to 2^-} f(x) = 1$

(b) $\lim_{x\to 2^+} f(x) = 3$

(c) $\lim_{x\to 2} f(x) =$ doesn't exist

(d) $f(2) = 3$

(e) $\lim_{x\to 3^-} f(x) = 1$

(f) $\lim_{x\to 3^+} f(x) = 1$

(g) $\lim_{x\to 3} f(x) = 1$

(h) $f(3) = 1$

(i) $\lim_{x\to 4^-} f(x) = 3$

(j) $\lim_{x\to 4^+} f(x) = 3$

(k) $\lim_{x\to 4} f(x) = 3$

(l) $f(4) = 1$

(m) $f'(3) = 3$

[10 points] 3. Compute the derivative $g'(3)$, where $g(x) = \sqrt{x+1}$, **using the definition of the derivative**. You will not receive credit for computing the derivative using differentiation rules. You must compute it directly using the limit definition of the derivative.

Solution: The difference quotient from $x = 3$ to $x = 3 + h$ is

$$\frac{g(3+h) - g(3)}{h} = \frac{\sqrt{h+4} - 2}{h}.$$

Now we need to manipulate the expression until we can cancel out a factor of h. We use the trick of multiplying the top and bottom of the fraction by the conjugate:

$$\begin{aligned} \frac{g(3+h) - g(3)}{h} &= \frac{\sqrt{h+4} - 2}{h}\left(\frac{\sqrt{h+4}+2}{\sqrt{h+4}+2}\right) \\ &= \frac{(h+4) - 4}{h\big(\sqrt{h+4}+2\big)} \\ &= \frac{h}{h\big(\sqrt{h+4}+2\big)} \\ &= \frac{1}{\sqrt{h+4}+2} \end{aligned}$$

So,

$$g'(3) = \lim_{h\to 0} \frac{g(3+h) - g(3)}{h} = \lim_{h\to 0} \frac{1}{\sqrt{h+4}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}.$$

[16 points] 4. Find the following derivatives.

(a) Let $f(x) = 3x^2 + 4x + 5$. Find $f'(x)$.

Solution:

$$f'(x) = 6x + 4.$$

(b) Compute $\dfrac{d}{dt}\left(te^t + \dfrac{1}{\sqrt{t}}\right)$.

Solution:

$$\begin{aligned}\frac{d}{dt}\left(te^t + \frac{1}{\sqrt{t}}\right) &= \frac{d}{dt}(te^t) + \frac{d}{dt}(t^{-1/2}) \\ &= e^t + te^t - \frac{1}{2}t^{-3/2}.\end{aligned}$$

(c) Compute $\dfrac{d}{dx}\left(\dfrac{x+1}{x^2+1}\right)$.

Solution:

$$\begin{aligned}\frac{d}{dx}\left(\frac{x+1}{x^2+1}\right) &= \frac{(x^2+1)\frac{d}{dt}(x+1)-(x+1)\frac{d}{dt}\big(x^2+1\big)}{(x^2+1)^2} \\ &= \frac{x^2+1-(x+1)(2x)}{(x^2+1)^2} \\ &= \frac{x^2+1-2x^2-2x}{(x^2+1)^2} = \frac{-x^2-2x+1}{(x^2+1)^2}\end{aligned}$$

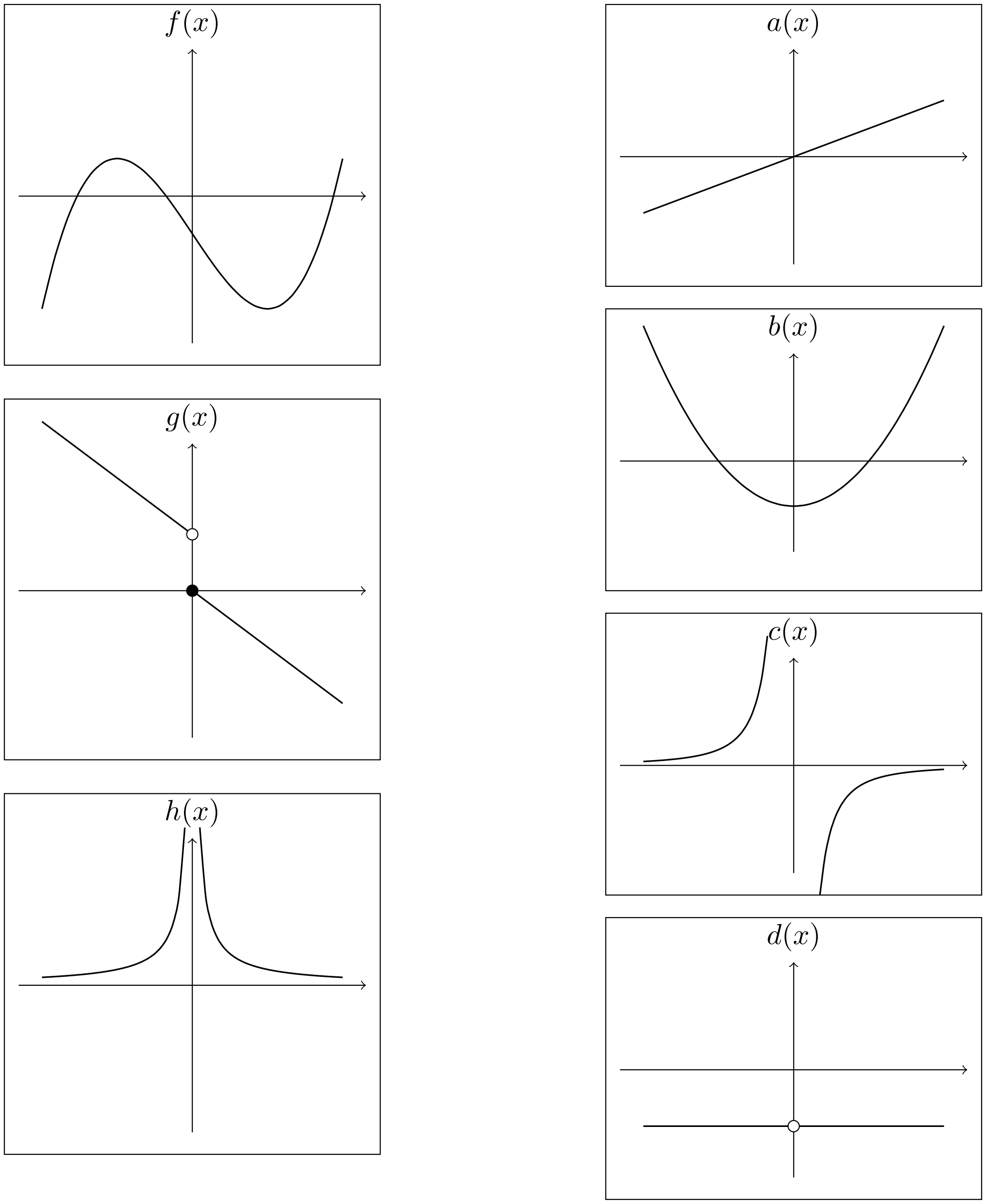
(d) Let $f(x) = \dfrac{5x^4+x}{x}$. Find $f'(x)$.

Solution: $f(x) = 5x^3+1$, except that $f(0)$ is undefined. And so

$$f'(x) = 15x^2,$$

with $f'(0)$ undefined (I will not take off points if you don't mention this).

[9 points] 5. On the left are three functions $f(x)$, $g(x)$, and $h(x)$. On the right are four functions, $a(x)$, $b(x)$, $c(x)$, and $d(x)$.



Circle one answer for each question. You do not need to explain yourself or show work.

(a) The derivative of $f(x)$ is... $a(x)$ **b(x)** $c(x)$ $d(x)$ none of the above

(b) The derivative of $g(x)$ is... $a(x)$ $b(x)$ $c(x)$ **d(x)** none of the above

(c) The derivative of $h(x)$ is... $a(x)$ $b(x)$ **c(x)** $d(x)$ none of the above

[10 points] 6. Let $g(x) = \dfrac{e^x}{x^2+1}$. Find all x-coordinates where the graph of this function has a horizontal tangent line.

Solution: First we use the quotient rule to compute

$$g'(x) = \frac{(x^2+1)e^x - e^x(2x)}{(x^2+1)^2} = \frac{e^x(x^2-2x+1)}{(x^2+1)^2} = \frac{e^x(x-1)^2}{(x^2+1)^2}.$$

The tangent line to the graph is horizontal at x-values where $g'(x) = 0$. So we must solve the equation

$$\frac{e^x(x-1)^2}{(x^2+1)^2} = 0.$$

Multiply both sides by $(x^2+1)^2$ (which is always nonzero):

$$e^x(x-1)^2 = 0.$$

This is equal to zero either when $e^x = 0$ or $x+1 = 0$. And e^x is never equal to 0. So the only solution is $x = 1$.

This page can be used as scratch paper.