**Math 231, Final, Version A** Name: ______________________

December 17, 2019

*I pledge that I have neither given nor received unauthorized assistance during this examination.*

**Signature:**

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise, **you must show your work** so that it's clear how you arrived at your answer.
- There are 10 problems on 12 pages.

| Question | Points | Score |
|---|---|---|
| 1 | 12 | |
| 2 | 12 | |
| 3 | 9 | |
| 4 | 10 | |
| 5 | 12 | |
| 6 | 7 | |
| 7 | 8 | |
| 8 | 10 | |
| 9 | 6 | |
| 10 | 14 | |
| Total: | 100 | |

**Good luck!**

[12 points] 1. Find the derivatives of the following functions. **Do not simplify your solutions.**

(a) $f(x) = 4\cos\big(2x^3 + 1\big)$

**Solution:**

$$f'(x) = -4\sin\big(2x^3 + 1\big)(6x^2)$$

(b) $g(u) = 1 + u^2 \ln u + \frac{1}{u}$

**Solution:**

$$g'(u) = 2u\ln u + \frac{u^2}{u} - \frac{1}{u^2}.$$

(c) $f(x) = e^{\sqrt{x+2}}$

**Solution:**

$$f'(x) = e^{\sqrt{x+2}}\frac{1}{2\sqrt{x+2}}$$

[12 points] 2. Compute the following integrals.

(a) $\displaystyle\int \left(3x^4 + \sqrt{x} + \frac{2}{x}\right) dx$

**Solution:**

$$\frac{3}{5}x^5 + \frac{2}{3}x^{3/2} + 2\ln|x| + C$$

(b) $\displaystyle\int_1^5 e^{-x}\, dx$

**Solution:**

$$-e^{-x}\Big|_1^5 = -e^{-5} + e^{-1}$$

(c) $\displaystyle\int \frac{(\ln x)^3}{x}\, dx$

**Solution:** Set $u = \ln x$. Then $du = \frac{1}{x}\, dx$, and

$$\int \frac{(\ln x)^3}{x}\, dx = \int u^3\, du = \frac{1}{4}u^4 + C = \frac{1}{4}(\ln x)^4 + C$$

[9 points] 3. Compute the following limits. If the limit does not exist, say so. Justify your answers by showing work or otherwise explaining your reasoning.

(a) $\lim_{x \to 1} \sqrt{x}$

**Solution:**

$$\lim_{x \to 1} \sqrt{x} = \sqrt{1} = 1.$$

(b) $\lim_{x \to \infty} \frac{x^2}{e^{2x}}$

**Solution:** By L'Hôpital's rule,

$$\lim_{x \to \infty} \frac{x^2}{e^{2x}} = \lim_{x \to \infty} \frac{2x}{2e^{2x}} = \lim_{x \to \infty} \frac{2}{4e^{2x}} = 0.$$

(c) $\lim_{x \to \infty} \frac{x}{x^2 - 4x + 4}$

**Solution:**

$$\lim_{x \to \infty} \frac{x}{x^2 - 4x + 4} = \lim_{x \to \infty} \frac{1}{x - 4 + 4/x} = 0.$$

[10 points] 4. A river runs through a field. You would like to fence off a rectangular region of the field by putting up three pieces of fence, using the river as the fourth side (see picture). Give the dimensions of the largest region you can make if you have 40 meters of fencing.


**Solution:** Let $x$ be the length of fence parallel to the river, and let $y$ be the lengths of the other two sides of the pen. Our goal is to maximize $xy$ subject to the constraint $x + 2y = 40$. Thus we want to maximize

$$f(y) = (40 - 2y)y = 40y - 2y^2$$

for $y$ in the interval $[0, 20]$. We compute

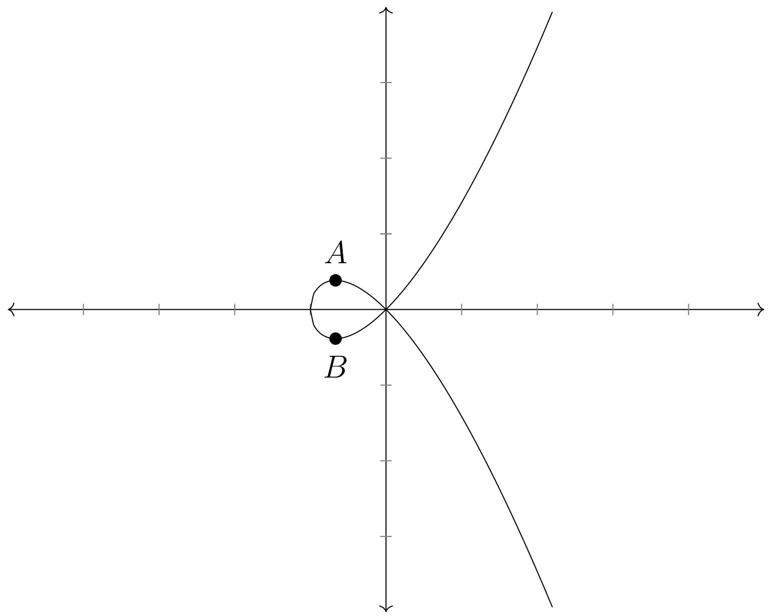
$$f'(y) = 40 - 4y.$$

Setting this equal to zero gives us the critical point $y = 10$. We check that

$$f(0) = 0,$$
$$f(10) = 200,$$
$$f(30) = 0.$$

So the optimal choice is $y = 10$, which gives us $x = 20$.

[12 points] 5. Consider the curve defined by $y^2 = x^2(x+1)$, shown below:




(a) Find the slope of the tangent line to the curve at the point $(3, 6)$.

**Solution:** Implicit differentiation gives

$$2yy' = 3x^2 + 2x,$$

yielding $y' = \frac{3x^2+2x}{2y}$. Plugging in $x = 3$, $y = 6$ gives:

$$y' = \frac{27+6}{12} = \frac{11}{4}.$$

(b) Give the equation of this tangent line.

**Solution:** Using point-slope form, $y - 6 = \frac{11}{4}(x - 3)$.

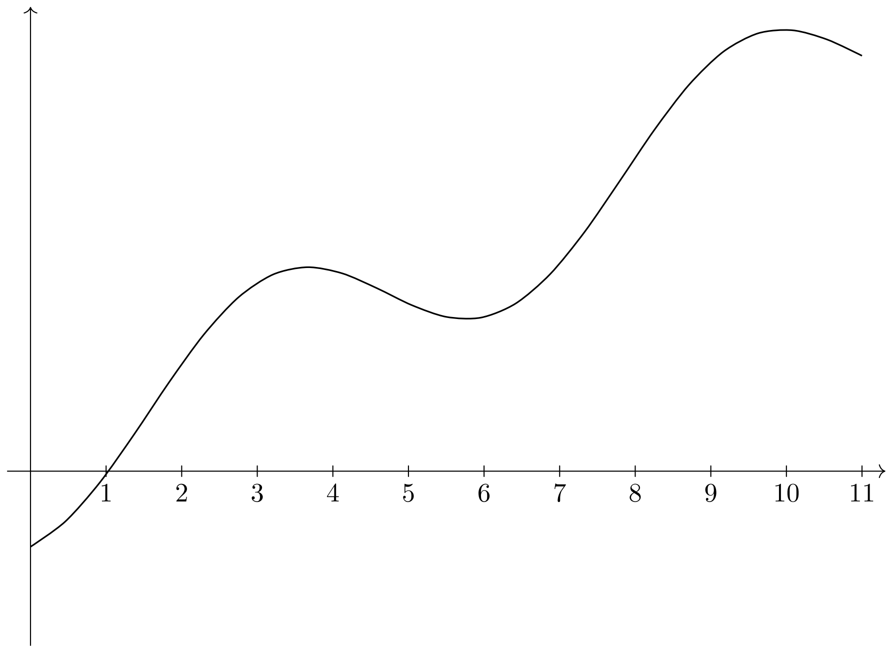
(c) Find the coordinates of the two points where the tangent line is horizontal (labeled $A$ and $B$ in the figure).

**Solution:** To solve $y' = 0$, we solve $3x^2 + 2x = x(3x + 2) = 0$ and get solutions $x = 0$ and $x = -2/3$. When $x = 0$, we have $y = 0$, and then $y'$ is undefined. But when $x = -2/3$, we get $y = \pm\frac{2}{\sqrt{27}}$, and at these points—$(-2/3, \frac{2}{\sqrt{27}})$ and $(-2/3, \frac{2}{\sqrt{27}})$, we have horizontal tangent lines.

(d) Give the equations of these two tangent lines.

**Solution:** $y = \frac{2}{\sqrt{27}}$ and $y = -\frac{2}{\sqrt{27}}$.

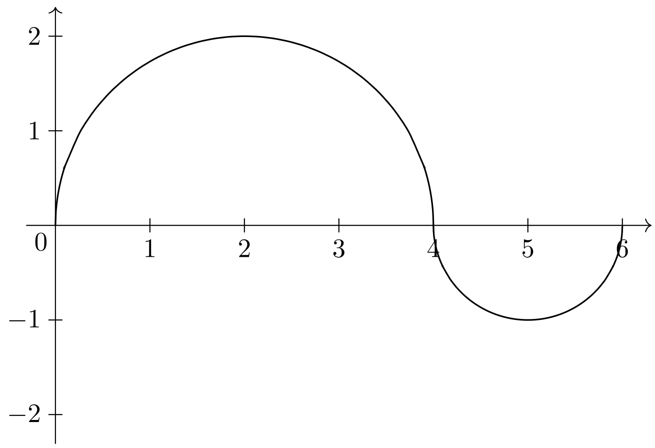
[7 points] 6. Here is a plot of the function $f(x)$:




Answer the following questions about the signs of $f'(x)$ and $f''(x)$. You do not need to justify or explain your answers.

| | | | |
|---|---|---|---|
| $f'(1)$ is: | ◯ negative | √ **positive** | ◯ zero |
| $f'(3)$ is: | ◯ negative | √ **positive** | ◯ zero |
| $f'(7)$ is: | ◯ negative | √ **positive** | ◯ zero |
| $f'(9)$ is: | ◯ negative | √ **positive** | ◯ zero |
| $f''(3)$ is: | √ **negative** | ◯ positive | ◯ zero |
| $f''(7)$ is: | ◯ negative | √ **positive** | ◯ zero |
| $f''(9)$ is: | √ **negative** | ◯ positive | ◯ zero |

[8 points] 7. Let $g(t)$ be the function whose graph is below. The two parts of the graph are semicircles.




Find $\int_0^6 g(t)\,dt$.

**Solution:** The integral is equal to the area of the first semicircle minus the area of the second, or

$$\pi(2)^2 - \pi(1)^2 = 3\pi.$$

[10 points] 8. A circular oil slick is growing at the rate of 8 square meters per second. How fast is the radius increasing when the radius is 4 meters?

**Solution:** Let $A$ be the area and $r$ the radius of the slick at time $t$.

**Know:**
$\frac{dA}{dt} = 10\text{m}^2/\text{s}$.

**Want:**
$\frac{dr}{dt}$ when $r = 4$

**Relation:**
$A = \pi r^2$

**Differentiating and solving:**
Differentiating both sides of the relation with respect to $t$ gives

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt},$$

giving us

$$\frac{dr}{dt} = \frac{dA}{dt}\frac{1}{2\pi r} = \frac{10}{2\pi(4)} = \frac{5}{4\pi}.$$

[6 points] 9. Give an estimate of $\sqrt{8.9}$ using linear approximation (also known as linearization).

**Solution:** Let $f(x) = \sqrt{x}$. Then $f'(x) = \frac{1}{2\sqrt{x}}$. So $f(9) = 3$ and $f'(9) = \frac{1}{6}$. We want to go from this known point $x = 9$ to $x = 8.9$ and estimate the new value of $f(x)$, which we can do with the linearization formula, applied here with $x = 9$ and $\delta x = -.1$:

$$f(8.9) = f(x + \delta x) \approx f(x) + f'(x)\delta x = 3 + \frac{1}{6}(-.1) = 2 + \frac{59}{60}.$$

[14 points] 10. Let $f(x) = \frac{1}{12}x(x-9)(x-24)$. The first and second derivatives of $f(x)$ are

$$f'(x) = \frac{1}{4}(x-4)(x-18)$$

and

$$f''(x) = \frac{1}{2}(x-11).$$

You do **not** need to check this differentiation.

(a) State all intervals where $f(x)$ is increasing. If there are none, write *none*.

**Solution:** $(-\infty, 4)$ and $(18, \infty)$

(b) State all intervals where $f(x)$ is concave up. If there are none, write *none*.

**Solution:** $(11, \infty)$

(c) Graph the function on the axes below. Be sure that all zeros, local maxima and minima, and inflection points are marked on the graph, and that their $x$-coordinates are given (you don't need to give their $y$-coordinates).