Math 231, Final, Version A Name: ____________________

May 22, 2018

I pledge that I have neither given nor received unauthorized assistance during this examination.

Signature:

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise, **you must show your work** sufficiently much that it's clear how you arrived at your answer.
- There are 10 problems on 11 pages.

Question	Points	Score
1	8	
2	8	
3	12	
4	9	
5	9	
6	11	
7	10	
8	11	
9	12	
10	10	
Total:	100	

Good luck!

[8 points] 1. Find the derivatives of the following functions. Please do **not** simplify your solutions.

(a) $f(x) = e^{x^3} \sin x$

Solution:

$$f'(x) = e^{x^3}(3x^2)\sin x + e^{x^3}\cos x$$

(b) $g(t) = \frac{1}{3t^2} - \sqrt{t} + \ln(4t+1) + 1$

Solution:

$$-\frac{2}{3t^3} - \frac{1}{2\sqrt{t}} + \frac{4}{4t+1}$$

[8 points] 2. Let $f(x) = \frac{x}{1+x^2}$. Find an equation for the tangent line to the graph of $f(x)$ at $x = 2$.

Solution: First, find $f'(x)$:

$$f'(x) = \frac{(1+x^2)(1) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}.$$

So, $f'(2) = \frac{1-4}{(1+4)^2} = -\frac{3}{25}$.

The line contains the point $(2, f(2))$ and has slope $-\frac{3}{25}$. In point-slope form, it's

$$y - \frac{2}{5} = -\frac{3}{25}(x-2).$$

In slope-intercept form, it's

$$y = -\frac{3}{25}x + \frac{16}{25}.$$

[12 points] 3. Compute the following integrals.

(a) $\int \left(x^2 + \frac{2}{x}\right) dx$

Solution:

$$\frac{x^3}{3} + 2\ln x + C$$

(b) $\int 2e^{\sin t} \cos t \, dt$

Solution: Set $u = \sin t$, so that $du = \cos t \, dt$.

$$\begin{aligned}\int 2e^{\sin t} \cos t \, dt = \int 2e^u \, du &= 2e^u + C \\ &= 2e^{\sin t} + C\end{aligned}$$

(c) $\int_0^2 3e^{-2x} \, dx$

Solution:

$$\int_0^2 3e^{-2x} \, dx = -\frac{3}{2}e^{-2x}\Big|_0^2 = -\frac{3}{2}\left(e^{-4} - 1\right).$$

[9 points] 4. Compute the following limits. If the limit does not exist, say so. Justify your answers by showing work or otherwise explaining your reasoning.

(a) $\lim_{x\to-\infty} \frac{3x^4 + x^3 - 100x^2 + 1}{5x^4 + 2}$

Solution:

$$\lim_{x\to-\infty} \frac{3x^4 + x^3 - 100x^2 + 1}{5x^4 + 2} = \lim_{x\to-\infty} \frac{3 + x^{-1} - 100x^{-2} + x^{-4}}{5 + 2x^{-4}}$$
$$= \frac{3+0+0+0}{5+0} = \frac{3}{5}$$

(b) $\lim_{x\to\infty} xe^{-2x}$

Solution:

$$\lim_{x\to\infty} xe^{-2x} = \lim_{x\to\infty} \frac{x}{e^{2x}}.$$

Since the top and bottom both diverge to infinity, we apply L'Hôpital's rule:

$$\lim_{x\to\infty} xe^{-2x} = \lim_{x\to\infty} \frac{1}{2e^{2x}}.$$

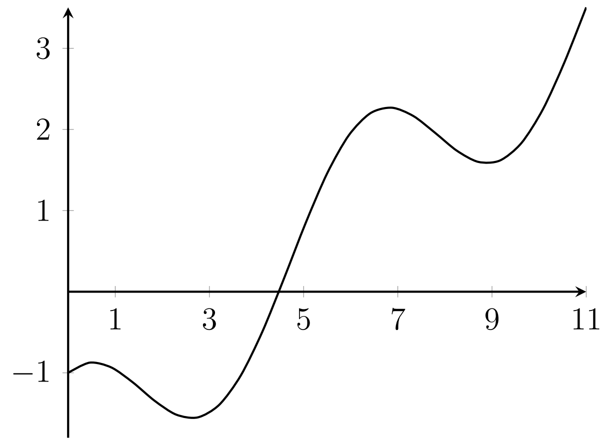
Since the bottom of the fraction diverges to infinity and the top is constant, the limit is equal to 0.

(c) $\lim_{x\to3} \frac{2\sin(x-3)}{\cos(x-3)}$

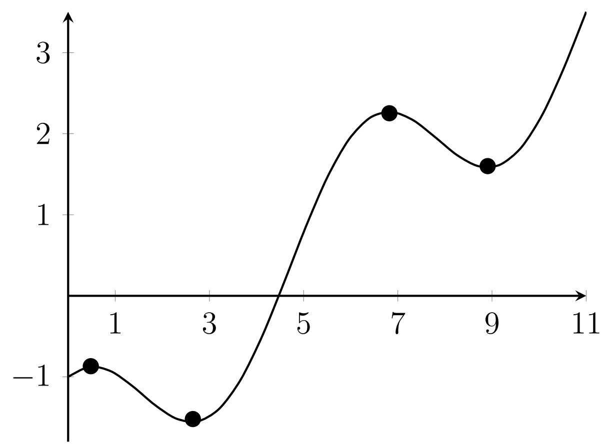
Solution: Since the function is continuous, we can compute the limit by evaluating it at $x = 3$, giving $\frac{2(0)}{1} = 0$.

[9 points] 5. This question asks about you to label certain points on a graph of a function with domain $[0, 11]$, shown in the problems below.

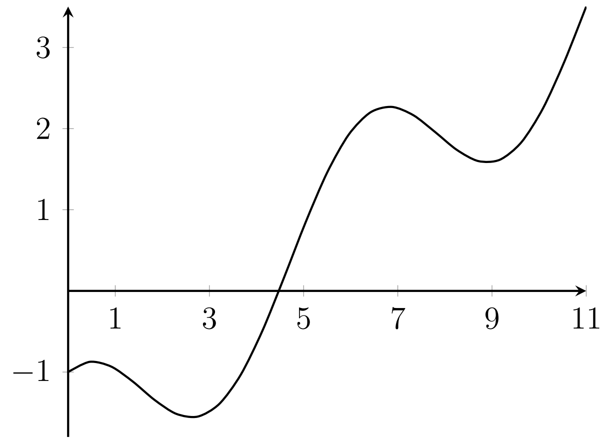
(a) Draw points on the graph at the (x, y) locations where **local** maxima or minima occur.



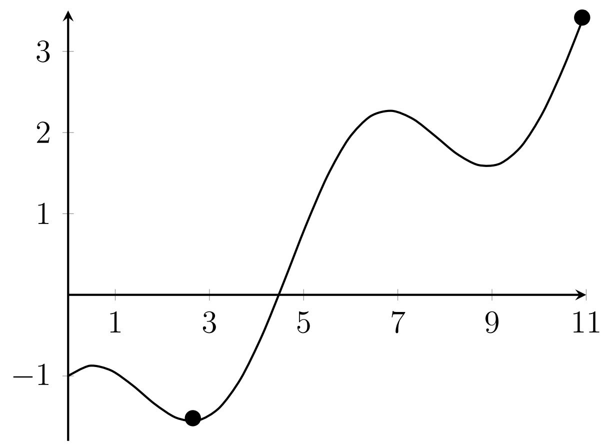
Solution:



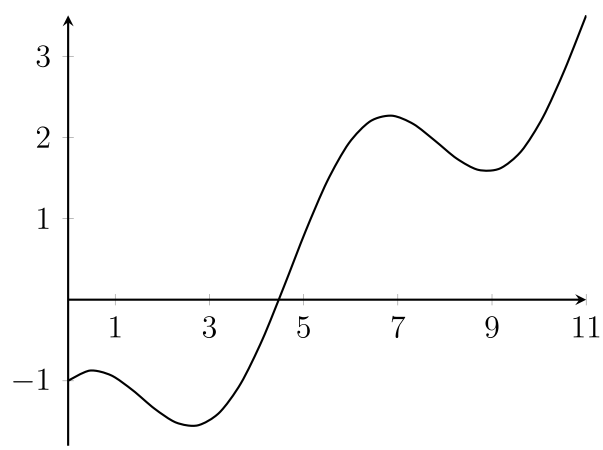
(b) Draw points on the graph at the (x, y) locations where **absolute** maxima or minima occur for the function on the domain $[0, 11]$.



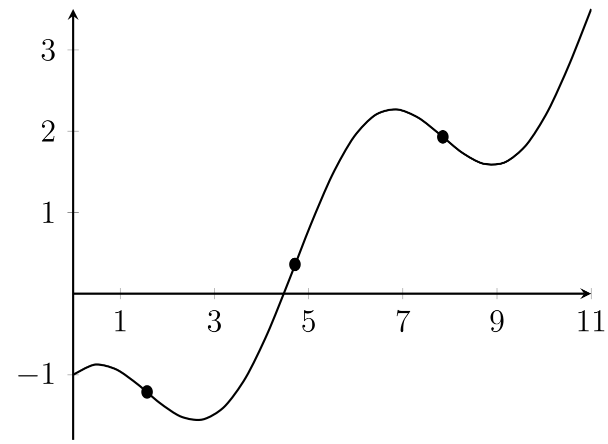
Solution:



(c) Draw points on the graph at the (x, y) locations where inflection points occur.



Solution:


3
2
1
−1
1
3
5
7
9
11

[11 points] 6. Find the two nonnegative numbers x and y that add up to 6 and make xy^2 as large as possible.

Solution: We want to maximize xy^2 subject to the constraint $x+y = 6$. Substituting $x = 6 - y$, we want to maximize $f(y) = (6 - y)y^2$ for y in $[0, 6]$. We find the critical points:

$$f'(x) = 12y - 3y^2 = 3y(4 - y) = 0.$$

So, there are critical points at $y = 0$ and $y = 4$. Evaluating at these and at the other endpoint $y = 6$,

$$f(0) = 0,$$
$$f(4) = 32,$$
$$f(6) = 0.$$

So, the maximizing choice of x and y is $x = 2$, $y = 4$.

[10 points] 7. A spherical balloon is filled with air at a rate of 2 cm^3 per minute. When the balloon has radius 8 cm, what is the rate of change of its radius? (The volume of a sphere with radius r is given by the formula $V = \frac{4}{3}\pi r^3$.)

Solution: We're given $\frac{dV}{dt} = 2$, and we'd like to find $\frac{dr}{dt}$ when $r = 8$. Differentiating both sides of the equation $V = \frac{4}{3}\pi r^3$,

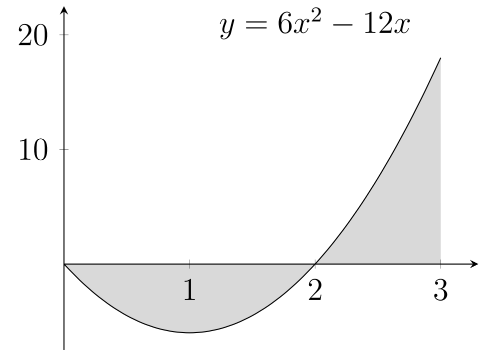
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

This gives us the equation

$$2 = 4\pi(64)\frac{dr}{dt}.$$

Solving this, we find that $\frac{dr}{dt} = \frac{1}{128\pi}$ cm/s.

[11 points] 8. Compute the total (unsigned) area between the graph of $f(x) = 6x^2 - 12x$ and the x-axis from $x = 0$ to $x = 3$. This is the shaded region in the graph sketched here.



Solution:

$$-\int_0^2 (6x^2 - 12x)\,dx + \int_2^3 (6x^2 - 12x)\,dx = -(2x^3 - 6x^2)\Big|_0^2 + (2x^3 - 6x^2)\Big|_2^3$$
$$= -(2(8) - 24) + (2(27) - 54) - (16 - 24)$$
$$= 8 + 0 + 8 = 16.$$

[12 points] 9. Let $f(x) = x^4 + 4x^3 - 16x + 1$. The first and second derivatives of f are

$$f'(x) = 4x^3 + 12x^2 - 16 = 4(x+2)^2(x-1)$$

and

$$f''(x) = 12x^2 + 24x = 12x(x+2).$$

(a) State all intervals where $f(x)$ is increasing. If there are none, write *none*.

Solution: $(1, \infty)$

(b) State all intervals where $f'(x)$ is concave up. If there are none, write *none*.

Solution: $(-\infty, -2)$, $(0, \infty)$

(c) Give all x-values where local minima occur. If there are none, write *none*.

Solution: $x = 1$

(d) Give all x-values where local maxima occur. If there are none, write *none*.

Solution: none

(e) Give all x-values where inflection points occur. If there are none, write *none*.

Solution: $x = -2, 0$

[10 points] 10. Consider the curve defined by the equation

$$e^{xy} = 1 + 2y.$$

Find $\dfrac{dy}{dx}$. Your answer will be in terms of both x and y.

Solution: Differentiating both sides of the equation,

$$e^{xy}\left(y + x\frac{dy}{dx}\right) = 2\frac{dy}{dx}.$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{ye^{xy}}{2 - xe^{xy}}.$$