1. Simplify:

(a) \[(x^{-1}y^2)^{-2} = (\frac{y^{-2}x^3}{x^{-2}y^2})^{-2} = (\frac{y^4}{x^4})^{-2} = \frac{x^8}{y^8}\]

Solution:

(b) \[\frac{\sqrt[3]{x^2}y^3}{\sqrt[5]{y^2x^5}}\]

Solution:

\[
\frac{\sqrt[3]{x^2}y^3}{\sqrt[5]{y^2x^5}} = \frac{x^{2/3}y^3}{|y|x^{1/5}} = \frac{y^3}{|y|x^{-13/3}}
\]

If it were assumed that \(y > 0\), then we could simplify \(y^3/|y|\) to \(y^2\).

2. Find the domain of the following expression:

\[\frac{x^2 + 1}{x^3 - 3x^2 - 10x}\]

Solution: The domain consists of all numbers that we can possibly plug into the expression, i.e., all numbers except for the ones that make the expression undefined. The only way this expression will be undefined is if the denominator of the fraction is 0. So, we rewrite it by factoring the denominator:

\[
\frac{x^2 + 1}{x(x^2 - 3x - 10)} = \frac{x^2 + 1}{x(x - 5)(x + 2)}.
\]

Now we can see that the expression is undefined when \(x = 0\), \(x = 5\), or \(x = -2\). So, the domain is all real numbers except for 0, 5, and -2.

3. Find all solutions to the following equations or inequalities:

(a) \(\sqrt{x} - x^2 = 0\)

Solution: We add \(x^2\) to both sides to get the equivalent equation \(\sqrt{x} = x^2\), and then we square both sides to get the equation

\[x = x^4.\]
(Note that this equation is not equivalent to the original one, in that squaring both sides of an equation might introduce extra solutions.) Subtract $x$ from both sides and factor to get

$$x(x^3 - 1) = 0.$$ 

The only solution to $x^3 - 1 = 0$ is $x = 1$. So the only two solutions to this equation are $x = 0$ and $x = 1$. Finally, we check that both of these solutions work in the original equation.

Your solution to a problem like this on an exam does not need to be as wordy as this. But it is important that if you do a step that can introduce extra solutions to an equation (e.g., squaring both sides of an equation or multiplying to clear a denominator), you check that all the solutions you find in the end work in the original equation.

(b) $3x^2 - 17x + 4 = 0$

**Solution:** Use the quadratic formula:

$$x = \frac{17 \pm \sqrt{241}}{6}$$

(c) $2x = x^2$

**Solution:** Put all terms on one side and factor:

$$x(x - 2) = 0$$

So, the solutions are $x = 0$ and $x = 2$.

(d) $(x - 1)(x - 2)(x - 3) > 0$

**Solution:** The way to solve an inequality like this is to work out on which intervals each of the factors is positive and negative:
(e) \(|x - 4| = 5\)

**Solution:** This equation is solved either when \(x - 4 = 5\) or \(x - 4 = -5\). The first equation gives the solution \(x = 9\) and the second gives \(x = -1\).

4. Give equations for lines fitting the following descriptions:

(a) line perpendicular to the line \(6x - 3y + 7 = 0\) and passing through point \((3, 1)\)

**Solution:** By rewriting \(6x - 3y + 7 = 0\) as \(y = 2x + \frac{7}{3}\), we see that it has slope 2. So the line we make should have slope \(-1/2\). Using the point-slope form of a line, the line with slope \(-1/2\) passing through points \((3, 1)\) is formed by the equation

\[y - 1 = -\frac{1}{2}(x - 3).\]

(b) line passing through points \((2, 0)\) and \((0, 4)\)

**Solution:** The slope of the line is \(\frac{4 - 0}{0 - 2} = -2\). The \(y\)-intercept of the line is 4 since the line includes the point \((0, 4)\), so its equation using the slope-intercept form is

\[y = -2x + 4.\]

(c) line parallel to the line \(2x - 6y = 3\) and passing through point \((0, 0)\)
Solution: By rewriting the given line as \( y = \frac{1}{3}x - \frac{1}{2} \), we see that it has slope \( \frac{1}{3} \). So, the line parallel to this with \( y \)-intercept 0 is

\[ y = \frac{1}{3}x. \]

5. The graph of \( y = f(x) \) is as shown.

![Graph of y = f(x)](image)

Sketch the graphs of the following functions:

1. \( y = f(x) - 2 \)  
2. \( y = f(x + 2) \)  
3. \( y = -f(x) \)  
4. \( y = 1 - f(x) \).

Solution:

1. shifted down by 2 from the original
2. shifted to the left by 2
3. flipped across the \( x \)-axis
4. flipped across the \( x \)-axis and then shifted up by 1

6. Put the following quadratic functions into standard form (i.e., express them as \( f(x) = (x - h)^2 + k^2 \)).

(a) \( x^2 - 10x + 22 \)  
(b) \( 3x^2 - 12x + 4 \)

Then for each parabola find:

- the vertex (the point at the bottom or top of the parabola)
- any \( x \)-intercepts (i.e., zeros/roots)
- the \( y \)-intercept
- the range of the function
Solution: For (a),

\[ x^2 - 10x + 22 = (x^2 - 10x + 25) - 25 + 22 = (x - 5)^2 - 3. \]

The vertex is then (5, −3). For the x-intercepts, you could either use the quadratic formula on the original equation or you could solve \((x - 5)^2 - 3 = 0\), which gives \(x = 5 \pm \sqrt{3}\). The y-intercept is 22 (just plug \(x = 0\) into the function). The range of the function is \([-3, \infty)\), since it’s an upward facing parabola with a minimum of −3 occurring at the vertex.

For (b),

\[ 3x^2 - 12x + 4 = 3(x^2 - 4x) + 4 = 3(x^2 - 4x + 4) - 12 + 4 = 3(x - 2)^2 - 8. \]

The vertex is (2, −8), the x-intercepts are \(x = 2 \pm \sqrt{8/3}\), the y-intercept is 4, and the range is \([-8, \infty)\).

7. A company sells bottles of hot sauce at a farmers market for $5 per jar. On average they sell 36 jars per week. They estimate that each ten cent decrease in price will increase sales by 2 jars per week.

(a) Find a function modeling weekly revenue in terms of the price.

Solution: If the price is \(x\), the number of ten cent decreases is \((5 - x)/.10\). So, the number of jars sold per week at price \(x\) is \(36 + 2(5 - x)/.10\). The total revenue is the number of jars times the price, or

\[ R(x) = x(36 + 2(5 - x)/.10) = x(36 + 20(5 - x)) = x(136 - 20x) = -20x^2 + 136x. \]

(b) Find the price to maximize weekly revenue.

Solution: We want to find the price that maximizes \(R(x)\). Since the graph of \(R(x)\) is a downward-facing parabola, we need to find its vertex. In standard form,

\[ R(x) = -20(x^2 - 6.8x) = -20(x^2 - 6.8x + 11.56) + 20(11.56) \]
\[ = -20(x - 3.4)^2 + 231.2 \]

The vertex occurs at price \(x = 3.4\), i.e., at price $3.40.

(c) According to the model, what price would be so high as to drive sales down to zero?
Solution: Using our formula $R(x) = -20x^2 + 136x$ and setting it equal to 0, we find that revenue is 0 when $x = 0$ or $x = 6.8$. So, sales are driven down to zero if we raise the price to $6.80 according to our model.

8. Find the following logarithms:
   (a) $\log_2 16$

   Solution: $\log_2 16$ is the solution to $2^x = 16$, which is 4.

   (b) $\log_9 \sqrt{3}$

   Solution: 1/4

   (c) $2^{\log_2 7}$

   Solution: $\log_2 7$ is the solution to $2^x = 7$. So, raising 2 to this power gives you 7, i.e., $2^{\log_2 7} = 7$.

9. Find $x$:
   (a) $\log_3 x = 2$

   Solution: 9

   (b) $\log_4 x = 2$

   Solution: 16

   (c) $\log_2 (2x - 1) = 3$

   Solution: Raise 2 to both sides of the equation to get
   $$2x - 1 = 2^3 = 8.$$ Solving this equation gives $x = 4.5$.

10. Use the laws for manipulating logarithms as specified:
    (a) (Expand) $\log_3(x\sqrt{yz^2})$
Solution:

\[
\log_3(x \sqrt{yz^2}) = \log_3 x + \log_3 \sqrt{y} + \log_3 (z^2) = \log_3 x + \frac{1}{2} \log_3 y + 2 \log_3 z.
\]

(b) (Combine) \(\ln(a + b) + \ln(a - b) - \ln c\)

Solution:

\[
\ln(a + b) + \ln(a - b) - \ln c = \ln \left( \frac{(a + b)(a - b)}{c} \right).
\]

(c) (Expand) \(\log \frac{x^2 y^3}{z^5 w^4}\)

Solution:

\[
\log \frac{x^2 y^3}{z^5 w^4} = 2 \log x + 3 \log y - 5 \log z - 4 \log w.
\]

11. Solve the following equations. Give exact answers (which will need to be in terms of logarithmic or exponential functions).

(a) \(e^{2x} = 7\)

Solution:

\[
x = \frac{\ln 7}{2}
\]

(b) \(5^x = 4^{x+1}\)

Solution: Take logarithms of both sides of the equation to get

\[
x \ln(5) = (x + 1) \ln 4.
\]

Now solve to get

\[
x = \frac{\ln 4}{\ln 5 - \ln 4}.
\]

(c) \(\log(x - 4) = 2\)
12. The half-life of strontium-90 is 28 years. How long will it take a 50-mg sample to decay to a mass of 32 mg?

**Solution:** The model for exponential decay starting at weight 50 with half-life 28 years is

\[ M(t) = 50 \left( 2 \right)^{-t/28}. \]

Setting this equal to 32 and then dividing both sides by 50 gives

\[ 2^{-t/28} = 0.64. \]

Taking logarithms of both sides,

\[ (-t/28) \ln 2 = \ln(0.64), \]

and solving for \( t \) gives

\[ t = -\frac{28 \ln(0.64)}{\ln 2}, \]

or 18.028 years.

13. In a particularly bad zombie outbreak in Freaktown, the population of zombies was 100,000 in 2050, and 300,000 in 2055. Assuming that the zombie population grows exponentially,

(a) Find a function that models the zombie population \( t \) years after 2050.

**Solution:** Let \( t \) denote the number of years past 2050. We model the population as

\[ P(t) = 1000000e^{rt}. \]

We’re given \( P(5) = 300000 \), which we use to solve for \( r \):

\[ 300000 = 1000000e^{5r}. \]

We get \( r = \frac{\ln(3)}{5} \approx .220 \). So, \( P(t) = 1000000e^{.220t}. \)

(b) Find the time require for the population to double.
Solution: Solve $100000e^{220t} = 200000$. This gives $t = \ln(2)/.220 = 3.151$ years.

(c) Predict the zombie population in 2075.

Solution:

$$P(25) = 100000e^{.220(25)} \approx 24,469,193.$$

14. Suppose $1000$ is invested in an account earning $8\%$ annual interest.

(a) How much money is in the account after two years if the money is compounded once per year?

Solution: $1000(1 + .08)^2 = 1166.4$ dollars.

(b) How much money is in the account after two years if the money is compounded monthly?

Solution: $1000\left(1 + \frac{.08}{12}\right)^{(12)(2)} = 1172.89$ dollars

(c) How much money is in the account after two years if the money is compounded continuously?

Solution: $1000e^{.08(2)} = 1173.51$ dollars

(d) If the money is compounded continuously, how long does it take for it to grow to $1500$?

Solution: Solve $1000e^{.08t} = 1500$. The solution is

$$t = \ln(3/2)/.08 = 5.07 \text{ years}.$$

15. (a) $7\pi/12$ radians is ________ degrees. Solution: 105

(b) $\pi$ radians is ________ degrees. Solution: 180

(c) $3\pi/2$ radians is ________ degrees. Solution: 270

(d) 2 radians is ________ degrees. Solution: 114.59

(e) 120 degrees is ________ radians. Solution: $\frac{2\pi}{3}$

(f) 333 degrees is ________ radians. Solution: $\frac{37\pi}{20}$
16. Give the exact values of the following trigonometric functions.
   (a) \( \tan(\pi) \) Solution: 0
   (b) \( \cos(3\pi/4) \) Solution: \(-\sqrt{2}/2\)
   (c) \( \csc(-\pi/6) \) Solution: \(-2\)
   (d) \( \sin(3\pi/2) \) Solution: \(-1\)

17. Let \( A = \sin(\pi/11) \). Express the following trigonometric functions in terms of \( A \). (For example, if the question asked you to give \( \csc(\pi/11) \), the answer would be \( 1/A \).)
   (a) \( \sin(10\pi/11) \) Solution: \( A \)
   (b) \( \sin(-\pi/11) \) Solution: \(-A\)
   (c) \( \sin(21\pi/11) \) Solution: \(-A\)
   (d) \( \cos(\pi/11) \) Solution: \( \sqrt{1-A^2} \)
   (e) \( \cot(12\pi/11) \) Solution: \( \sqrt{1-A^2}/A \)

18. From the top of a 200-foot lighthouse, the angle of depression down to a ship is 19 degrees (i.e., from the lighthouse you must tilt your head downward 19 degrees to see the ship). How far is the ship from the base of the lighthouse?

   **Solution:** Draw a picture! You’ll see that if \( x \) is the distance we’re trying to find, then \( \tan(19^\circ) = \frac{200}{x} \). So, \( x = \frac{200}{\tan(19^\circ)} = 580.84 \) feet.

19. A right triangle has an angle of 35 degrees. If the hypotenuse has length 15, what are the lengths of the other two sides?

   **Solution:** Draw a picture! If the two sides have lengths \( x \) and \( y \), one will satisfy \( \cos(35^\circ) = x/15 \), and one will satisfy \( \sin(35^\circ) = x/15 \). So, the two lengths are \( x = 15 \cos(35^\circ) \approx 12.29 \) and \( y = 15 \sin(35^\circ) \approx 8.60 \).

20. Give all solutions for \( \theta \) between 0 and \( 2\pi \) to the following equations:
   (a) \( \sin \theta = .9 \)
      **Solution:** Draw a picture! There are solutions in the first and second quadrants. The one in the first quadrant is given by \( \sin^{-1}(0.9) \approx 1.120 \) radians. The one in the second quadrant is given by flipping the angle across the y-axis, so it’s \( \pi - \sin^{-1}(0.9) \approx 2.022 \) radians.
   (b) \( \cos \theta = -.2 \)
Solution: Draw a picture! There are solutions in the second and third quadrants. The one in the second quadrant is given by \( \cos^{-1}(-.2) = 1.772 \) radians. The one in the third quadrant is given by flipping this angle across the \( x \)-axis, which makes it \( 2\pi - \cos^{-1}(-.2) \approx 4.511 \) radians.

21. Walking in straight lines in a forest with no change in elevation, a hiker starts at point A and walks 1.2 km to point B. Then she walks 1.8 km to point C. Finally she returns to point A, going a distance of 0.8 km.

Standing at point A, she looks out at points B and C and measures the angle between them. What is it?

Solution: The hiker forms a triangle with vertices \( A \), \( B \), and \( C \) and side lengths \( a = 1.8 \), \( b = 0.8 \), and \( c = 1.2 \) where the side of length \( a \) is opposite vertex \( A \), the side of length \( b \) is opposite vertex \( B \), and the side of length \( c \) is opposite vertex \( C \). Our goal is to find the angle of the triangle at \( A \). The law of cosines tells us that

\[
b^2 + c^2 - 2bc \cos A = a^2.
\]

So,

\[
(0.8)^2 + (1.2)^2 - 2(0.8)(1.2) \cos A = (1.8)^2.
\]

This gives

\[
\cos A = -0.604167
\]

The solution to this equation between 0 and \( \pi \) (i.e., between 0 and 180 degrees, which is the range of possible angles in a triangle) is

\[
A = \cos^{-1}(-0.604167) = 2.2195 \text{ radians} = 127.17 \text{ degrees}.
\]

22. Suppose a triangle has angles of 35 and 45 degrees and the side opposite the 45-degree angle has length 2. What are the lengths of the other two sides?

Solution: Draw a picture!

The other angle in the triangle is 100 degrees. Let \( a \) be the side across from the 35-degree angle and \( b \) the side across from the 100-degree angle. The law of sines yields

\[
\frac{\sin(45^\circ)}{2} = \frac{\sin(35^\circ)}{a}
\]
and
\[ \frac{\sin(45^\circ)}{2} = \frac{\sin(100^\circ)}{b}. \]

Solving the first equation gives
\[ a = \frac{2 \sin(35^\circ)}{\sin(45^\circ)} \approx 1.622 \]

and the second gives
\[ b = \frac{2 \sin(100^\circ)}{\sin(45^\circ)} \approx 2.785 \]