Here are solutions to some of Prof. Kofman's 2013 midterm 2. By the way, you shouldn't bother with problems 1, 3, 4, and 5, which cover topics that we no longer include in the class.

Problem 2. Say that the garden has dimensions $x \times y$. It has three fences of length x—the outside of the garden perpendicular to the barn, and the fence down the middle splitting the garden, and then it has one fence of length y parallel to the barn.

So, the perimeter of the garden is $3x + y$ and the area is xy. Since we have 180 feet of fencing, we have $3x + y = 180$ and can write y as $y = 180 - 3x$. Now the area is

$$A(x) = xy = x(180 - 3x) = -3x^2 + 180x.$$

We complete the square to put this into standard form:

$$\begin{aligned} A(x) &= -3(x^2 - 60x) = -3(x^2 - 60x + 900 - 900) = -3(x^2 - 60x + 900) + 2700 \\ &= -3(x - 30)^2 + 2700. \end{aligned}$$

The maximum of $A(x)$ is 2700 and is found at $x = 30$. So, the largest garden we can make is 2700 square feet. One of its dimensions is $x = 30$ feet. The other is $y = 180 - 3x = 90$ feet.

Problem 6. Do the following without a calculator:

(a) $\log_6 72 + \log_6 3 = \log_6(72 \cdot 3) = \log_6(216)$. And this is equal to 3, since $6^3 = 216$.

(b) $\log_{27} 9 = \log_{27}(3^2) = 2\log_{27} 3$. Since 3 is the cube root of 27, we have $\log_{27} 3 = \frac{1}{3}$. So we get $\log_{27} 9 = \frac{2}{3}$.

(c) $\ln \frac{\sqrt[3]{e}}{e^4} = \ln \sqrt[3]{e} - \ln(e^4) = \frac{1}{3} - 4 = -11/3$

Problem 7.

$$\begin{aligned} \ln(3x + 5) + 2\ln(x^3 - 1) - \frac{2}{3}\ln(4x - 7) &= \ln(3x + 5) + \ln((x^3 - 1)^2) - \ln((4x - 7)^{2/3}) \\ &= \ln \frac{(3x + 5)(x^3 - 1)^2}{(4x - 7)^{2/3}} \end{aligned}$$

Problem 8. If $\ln a = -5$, $\ln b = 7$, $\ln c = -4$, evaluate the following expressions:

(a)

$$\begin{aligned} \ln \frac{b^3}{a^2 c^4} &= \ln(b^3) - \ln(a^2 c^4) \\ &= 3\ln b - (\ln(a^2) + \ln(c^4)) \\ &= 3\ln b - 2\ln a - 4\ln c = 3(7) - 2(-5) - 4(-4) = 47 \end{aligned}$$

(b)

$$\begin{aligned}
\ln(b\sqrt[3]{ac}) &= \ln(ba^{1/3}c^{1/3}) \\
&= \ln b + \frac{1}{3}\ln a + \frac{1}{3}\ln c \\
&= 7 + \frac{1}{3}(-5) + \frac{1}{3}(-4) = 4
\end{aligned}$$

(c) $\ln(e^3/b) = \ln(e^3) - \ln b = 3 - 7 = -4$

Problem 9.

(a) $4^{2x+1} = 5^{3x}$

Take natural logs of both sides:

$$(2x+1)\ln 4 = 3x\ln 5.$$

Expand and add/subtract from both sides to rearrange terms, then factor:

$$\begin{aligned}
2x\ln 4 + \ln 4 &= 3x\ln 5 \\
2x\ln 4 - 3x\ln 5 &= -\ln 4 \\
x(2\ln 4 - 3\ln 5) &= -\ln 4
\end{aligned}$$

So,

$$x = \frac{-\ln 4}{2\ln 4 - 3\ln 5} = 0.67435$$

(b) $\log_2(x-2) + \log_2(x+1) = 2$

Combine the logarithms:

$$\log_2((x-2)(x+1)) = 2$$

In exponential form, this equation is

$$2^2 = (x-2)(x+1).$$

Now we expand, add/subtract to get all terms on the same side, and then factor:

$$\begin{aligned}
4 &= x^2 - x - 2 \\
0 &= x^2 - x - 6 = (x-3)(x+2).
\end{aligned}$$

So, the potential solutions are $x = 3$ and $x = -2$. However, $x = -2$ can't be plugged into $\log_2(x-2)$ or $\log_2(x+1)$ since you can only take the logarithm of a positive number. So only $x = 3$ works.

(c) $3\ln(5-x) = 2$

Divide both sides to get $\ln(5-x) = \frac{2}{3}$. In exponential form, this gives

$$e^{2/3} = 5 - x.$$

So, $x = 5 - e^{2/3}$.