Problem 1

Consider vectors in a Hilbert space $\mathcal{H}$.

1. Show that an orthonormal set is linearly independent.
2. Show that $|(a,b)| = \|a\|\|b\|$ if and only if a and b are parallel. (Hint: Use a projection).

Problem 2

Consider the following differential operator L in $[0,\infty)$

$$Lf = -\mathrm{e}^{x}\frac{d}{dx}\left(x\mathrm{e}^{-x}\frac{df}{dx}\right)$$

1. Show by direct calculation that L is self-adjoint with respect to the inner product with weight $w(x) = \exp(-x)$. What do you need to assume about boundary conditions?
2. Consider $f(x) = -2x + 1$ and $g(x) = x^2 - 4x + 2$. Are f and g eigenfunctions of L?
3. Are $h(x) = 1$ and g orthogonal?

Problem 3

Consider the system given by

$$\begin{aligned}\dot{x} &= \mathrm{e}^{x} + \sin(5y) - \cos(2y)\\ \dot{y} &= x + 2\sin y\end{aligned}$$

Show that $(0,0)$ is a point of equilibrium and analyze the stability of the system around this point.

Problem 4

Consider the initial value problem

$$\dot{x} = \cos(t)x - \epsilon x^2, \qquad x(0) = 1$$

1. Find the solution of the problem for $\epsilon = 0$.
2. Introduction a slow time-scale $t_1 = \epsilon t$ and find the equation governing the slow-scale evolution of the solution. Note: Your solution may contain a definite integral that you do not need to solve.