1 Adams-Bashforth integration

A simple method to integrate an ordinary differential equation is the second-order Adams-Bashforth integration method. We assume an initial value problem of the form

$$\dot{y} = f(t, y(t)), \qquad y(0) = y_0 \,. \tag{1}$$

If $t_{n+1} = t_n + \Delta t$, then the second-order Adams-Bashforth method is the following discretization of (1):

$$y_{n+2} = y_{n+1} + \Delta t \left(\frac{3}{2} f(t_{n+1}, y_{n+1}) - \frac{1}{2} f(t_n, y_n) \right) . \tag{2}$$

Hence, the method assumes that y_n and y_{n+1} are known in order to compute y_{n+2}. This formula is derived in the following way: Clearly we have

$$y_{n+2} = y_{n+1} + \int_{t_{n+1}}^{t_{n+2}} \dot{y}(t) dt \tag{3}$$

The key idea is to use an approximation of $\dot{y}$ in order to approximate the above integral. For simplicity of notation, set $w(t) = \dot{y}(t)$. Clearly, we know the value of w at t_n and t_{n+1} and the simple (Lagrange) linear interpolation yields

$$w(t) \approx w_{n+1} \frac{t - t_n}{t_{n+1} - t_n} + w_n \frac{t - t_{n+1}}{t_n - t_{n+1}} \tag{4}$$

Assuming equidistant spacing of the t_n, hence $t_{n+1} - t_n = \Delta t$ and $t_{n+2} - t_n = 2\Delta t$, etc. Then it is easy to see that we obtain for the above integral

$$\begin{aligned} \int_{t_{n+1}}^{t_{n+2}} \dot{y}(t) dt &\approx \int_{t_{n+1}}^{t_{n+2}} \left(w_{n+1} \frac{t - t_n}{\Delta t} - w_n \frac{t - t_{n+1}}{\Delta t} \right) dt \\ &= \frac{3}{2} \Delta t w_{n+1} - \frac{1}{2} \Delta t w_n \\ &= \Delta t \left(\frac{3}{2} f(t_{n+1}, y_{n+1}) - \frac{1}{2} f(t_n, y_n) \right) . \end{aligned}$$

2 Application to nonlinear PDEs

A typical situation is to numerically solve a partial differential equation in which both a linear operator L and a nonlinear operator R are present, hence a PDE of the form

$$u_t = Lu + R(t, u), \qquad u(0) = u_0 \tag{5}$$

We assume that the linear operator L does not depend explicitly on the time t. Applying the Adams-Bashforth integration scheme to such a problem, we obtain the method

$$u_{n+2} = \mathrm{e}^{L\Delta t}\left(u_n + \Delta t\left(\frac{3}{2}R_{n+1} - \frac{1}{2}\mathrm{e}^{L\Delta t}R_n\right)\right) . \tag{6}$$

This follows more or less directly from the previous section, but it is worthwhile to look at the (quick) derivation: Using an integrating factor, we find the following integral equation equivalent to (5):

$$u(t_2) = \mathrm{e}^{L(t_2-t_1)}\left(u(t_1) + \int_{t_1}^{t_2} \mathrm{e}^{-L(s-t_1)}\, R(s,u(s))\, ds\right) \tag{7}$$

which is valid for any $t_2 > t_1$. Recalling the results from the previous section, we set $t_2 = t_{n+2}$ and $t_1 = t_{n+1}$ and approximate the integral as

$$\int_{t_{n+1}}^{t_{n+2}} \mathrm{e}^{-Ls}\, R(s,u(s))\, ds \approx \Delta t\left(\frac{3}{2}R_{n+1} - \frac{1}{2}\mathrm{e}^{L\Delta t}R_n\right)$$

3 Two-dimensional Navier-Stokes equations

Consider the two-dimensional Navier-Stokes equations in the form

$$u_t + (u\cdot\nabla)\, u = \kappa\Delta u - \nabla p, \qquad \nabla\cdot u = 0 \tag{8}$$

There are different ways to solve these equations numerically. A main point is how to enforce the constraint that the solution u has to be divergence-free.

3.1 Vorticity formulation

One possibility is the so-called *vorticity formulation*. Define the vorticity ω by

$$\nabla\times u = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u_{2x} - u_{1y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \tag{9}$$

Taking the curl of (8), after some algebra (exercise), we obtain a simple advection-diffusion equation for the vorticity ω:

$$\omega_t + (u\cdot\nabla)\,\omega = \kappa\Delta\omega \tag{10}$$

Note that this equation is fundamentally different in three dimensions due to the presence of an additional term (the vorticity stretching term). In two spatial dimensions, the vorticity is simply advected. Here, we concentrate on the much simpler two dimensional case.

In equation (10), u depends on ω and, in fact, we can compute u from ω using the Biot-Savart law. Since u is divergence-free, we have $\nabla \times \omega = \nabla \times \nabla \times u = -\Delta u$, hence

$$u = -\Delta^{-1}(\nabla \times \omega) = \begin{pmatrix} -\Delta^{-1}\partial_y \omega \\ \Delta^{-1}\partial_x \omega \\ 0 \end{pmatrix} \tag{11}$$

In our implementation (periodic boundary conditions, mean-free flow) we can easily compute all derivatives and Δ^{-1} in Fourier domain.

3.2 Elimination of the pressure term

Take the divergence of (8) in order to obtain a fundamental relationship between the nonlinear advection term and the pressure:

$$\nabla \cdot ((u \cdot \nabla)u) = -\Delta p \tag{12}$$

It is easy to see (exercise) that then we can solve for p and obtain

$$p = -\Delta^{-1}(u_{1x}^2 + 2u_{2x}u_{1y} + u_{2y}^2) \tag{13}$$

Note that this equation is sometimes rewritten by introducing the matrix ∇u defined as

$$\nabla u = \begin{pmatrix} u_{1x} & u_{1y} \\ u_{2x} & u_{2y} \end{pmatrix},$$

from which follows the trace formula

$$\mathrm{tr}((\nabla u)^2) = u_{1x}^2 + 2u_{2x}u_{1y} + u_{2y}^2 \tag{14}$$

With this notation, we can rewrite the original Navier-Stokes equations in pressure-free form

$$u_t + (u \cdot \nabla)\, u = \kappa \Delta u + \nabla \Delta^{-1} \mathrm{tr}((\nabla u)^2) \tag{15}$$

3.3 Leray formulation

We can also use the above relationship between nonlinearity and pressure in its original form

$$p = -\Delta^{-1}(\nabla \cdot ((u \cdot \nabla)u))$$

and eliminate the pressure term by writing

$$u_t + \left(1 - \nabla\Delta^{-1}\nabla\cdot\right)\left((u\cdot\nabla)u\right) = \kappa\Delta u \tag{16}$$

The operator P defined by

$$P = 1 - \nabla\Delta^{-1}\nabla\cdot \tag{17}$$

projects a vector field onto its divergence-free part. This follows by direct calculation from

$$\nabla\cdot(Pv) = \nabla\cdot v - \nabla\cdot\left(\nabla\Delta^{-1}\nabla\cdot v\right) = 0$$

If we using the Hodge-decomposition by writing $w = Pw + \nabla\phi$ we can consider an equation for the field w given by

$$w_t + ((Pw)\cdot\nabla)(Pw) = \kappa\Delta w \tag{18}$$

which is equivalent to Navier-Stokes if we set $u = Pw$. Note that the field w is in general compressible. The physical pressure is then given by $p = \phi_t - \kappa\Delta\phi$.